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ON CHOICE OF FUNCTIONAL FOR A VARIATIONAL PROBLEM OF GAS DYNAMICS

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Numerical solution of variational problem on construction of supersonic nozzle with uniform output flow is considered. The way of choice of minimizing functional is proposed. Comparison with the results obtained by another method is carried out.

1. Introduction

It is known that if numerical (programmed) solution of some problem is possible, this solution could be performed by not unique method. In this sense direct numerical methods of solving variational problems are no exception, when solution is obtained as a result of minimization of a definite functional. In this case for the class of variational problems including gas-dynamic problems some numerical realizations can result in the fact that definition domain is a holey set.

As a typical example let us consider the problem on numerical construction of supersonic nozzle with uniform output flow solved by direct method. It should be noted that the problem on nozzle of maximum draft is close to it [1]. Let flow field be calculated in some nozzle, the profile of which is defined by varied variables, but the functional characterising flow nonuniformity at nozzle output and having minimum value when the flow is uniform is calculated by the found field.

The main two approaches can be proposed as a base for numerical solution of the given problem. The former uses the fact that the flow keeps to be supersonic and it is possible to apply cruise schemes which are simple in realization and quick in calculation; the second approach supposes existence of subsonic flows and, hence, requires application of more complicated and slower numerical methods. Let the method of characteristics serve as a base of the first numerical solution [2]. It is evidently, then, that if in some contour supersonic flow would not be realized completely, and it would result in emergency stop. The consequence of it would be holey definition domain, as the functional using parameters of flow at nozzle output cannot be calculated. The second approach taking into account appearance of subsonic flow and using, foe example, pseudoviscosity method and Godunov's scheme [3] has no such disadvantage, however, intellectual and time consumption increases by the order.

In the work [4] along with method of characteristics permitting to get the solution quickly, the functional which particularly uses magnitudes of flow parameters found at each characteristic C^+ is proposed. It makes possible to proceed with the functional onto simply connected domain.

In the given article, which is development of the work [4], the functional is proposed to calculate by a simpler method. To estimate the accuracy of the results obtained the solution obtained by the method [2] was used.

2. The problem

It is given steady nonswirling isoentropic and isoenergetic flow of ideal perfect gas in axial-symmetric nozzle, fig. 1. Characteristic equations and compatibility conditions have the view:

$$\frac{dy}{dx} = \mathrm{tg}(\theta \pm \alpha),$$

$$d\theta \pm \frac{\cos^2(\alpha)}{(\gamma+1)/2 - \cos^2(\alpha)} d\alpha \pm \frac{\sin(\alpha)\sin(\theta)}{y\cos(\theta \pm \alpha)} dx = 0, \quad (1)$$

where γ – adiabatic exponent, further γ=1,4; x and y – longitudinal and lateral coordinates referred to nozzle radius of minimum section; α=arcsin $(1/M)$ – Mach's angle, M – Mach's number; θ – inclination of velocity vector (current line) to x axes; sign +(-) corresponds to characteristic of $C^+(C^-)$; it is supposed that in the input section $x_a=0$. On the wall the non-leaking condition is realized: $\mathrm{tg}(\theta)=f'(x)$, where $f(x)$ – function describing nozzle contour, the prime means derivative with respect to x. In the input section x_a flow is homogeneous: $\theta_{in}=0$, $M_{in}=1$, where index «*in*» corresponds to input of nozzle. Further index «0» corresponds to output of nozzle.

Calculating flow parameters at the angular point «*a*» Prandtl-Mayer formula is used, following from (1):

$$\theta_2+\psi(\alpha_2)=\theta_1+\psi(\alpha_1),$$

$$\psi(\alpha)=-\alpha+\sqrt{\frac{\gamma+1}{\gamma-1}}\,\mathrm{arctg}\left(\sqrt{\frac{\gamma+1}{\gamma-1}}\,\mathrm{tg}(\alpha)\right), \qquad (2)$$

where lower indexes 1 and 2 correspond to the parameters before and after turn at the angular point; $\theta_1=0$, $\alpha_1=\alpha_{in}$.

It is necessary to find nozzle contour (fig. 1), at the output of which in the section $x=x_b$ the flow must have homogeneous across the nozzle parameters ?=0 and Mach's number $M_0>1$. The pressure at the output is considered to be more than the pressure of the environment.

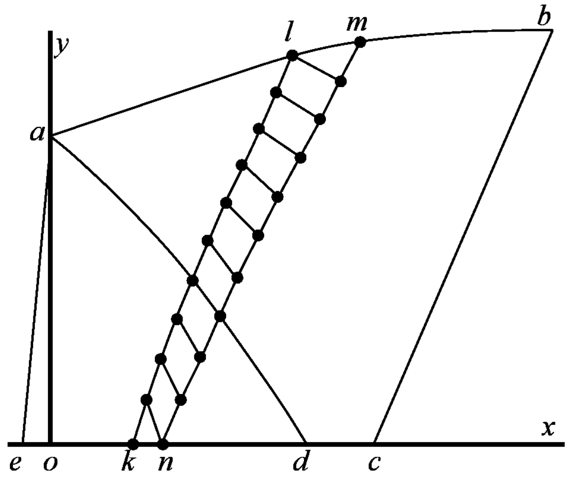

Fig. 1. *Scheme of nozzle. oa – minimum section; ab – nozzle with angular point; ad – characteristic of C^-, belonging to beam of rarefaction wave; ae – initial characteristic of ; kl – characteristic of ; nm – characteristic of ; cb – final characteristic of C^+*

3. Test problem

At the assigned adiabatic exponent let us consider one-parameter, depending on M_0 family of nozzles with angular point and homogeneous characteristic at the output. It is known [1] that these nozzles have maximum draft and are shortest and providing zero losses at the distance in the output section. As at the output the flow is homogeneous and parallel, the indicated nozzles will satisfy the conditions stated above.

Application of the method [2] at the assigned $M_0>1$ and assigned adiabatic exponent permits to find the only nozzle belonging to the family stated and satisfying the problem conditions. Nozzle contour obtained in this way is considered to be a reference and it will be compared with direct method, but the coordinates of the initial point «*a*» and the stated final point «*b*» will be given for direct method.

4. Direct method

The coordinates of the points «*a*», «*b*» and tangent of inclination angle at point «*b*» are given:

$$f(x_a)=y_a,\quad f(x_b)=y_b,\quad f'(x_b)=y'_b,$$
$$x_a=0,\quad y_a=1,\quad y'_b=0. \qquad (3)$$

The last equality follows from the condition $\theta=0$ at the nozzle output.

For approximation of sought contour the power polynomials are used as basic functions:

$$f(x)=\sum_{i=1}^{N+3}c_i t^{i-1},\ t=(2x-x_b-x_a)/(x_b-x_a),$$
$$t\in[-1,1],\ x\in[x_a,x_b]. \qquad (4)$$

As the conditions (3) should be met, the coefficients c_{N+1}, c_{N+2}, c_{N+3} and nozzle profile $f(x)$ are expressed through lineally dependent coefficients $c_1,\ldots,c_N$. It is required to find nozzle profile satisfying the geometrical conditions (3) and problem definition.

Direct calculation. In the context of the direct method a unit calculation of flow field is accepted as a direct calculation which is finished by functional calculation. The choice of the functional will be discussed below. In the direct calculation set of equations (1) is solved by the scheme shown in fig. 2, where the points with known parameters are denoted by figures «1» and «2», from which the characteristics C^+ and C^- follow. The point of their meeting is denoted by figure «3». Writing down the equations (1) in the difference form, we obtain the set of equations with respect to unknown parameters α_3^j, θ_3^j, x_3^j, y_3^j at point «3»:

$$\begin{cases}\dfrac{y_3^j-y_2}{x_3^j-x_2}=\mathrm{tg}(\theta_{23}-\alpha_{23}),\\[2mm] \dfrac{y_3^j-y_1}{x_3^j-x_1}=\mathrm{tg}(\theta_{13}+\alpha_{13}),\\[2mm] \theta_3^j-\theta_1+\dfrac{\cos^2\alpha_{13}}{(\gamma+1)/2-\cos^2\alpha_{13}}(\alpha_3^j-\alpha_1)+\\[2mm] +\dfrac{\sin\alpha_{13}\sin\theta_{13}}{y_{13}\cos(\theta_{13}+\alpha_{13})}(x_3^j-x_1)=0,\\[2mm] \theta_3^j-\theta_2-\dfrac{\cos^2\alpha_{23}}{(\gamma+1)/2-\cos^2\alpha_{23}}(\alpha_3^j-\alpha_2)-\\[2mm] -\dfrac{\sin\alpha_{23}\sin\theta_{23}}{y_{23}\cos(\theta_{23}-\alpha_{23})}(x_3^j-x_2)=0,\end{cases}$$

here $j=1,2,\ldots$ – iteration number. Denote $p=\alpha,\theta,x,y$, then $p_{13}=(p_1+p_3^{j-1})/2$, $p_{23}=(p_2+p_3^{j-1})/2$. The system obtained was solved by iteration up to meeting the condition $\max|p_3^j-p_3^{j-1}|<10^{-8}$. At the initial iteration was suggested

$p_3^0=(p_1+p_2)/2$. As on the axis $\theta=0$, $y=0$, but on the wall $\theta=\mathrm{arctg}(f'(x))$ and $y=f(x)$, then in the given set of equations evident simplifications were made.

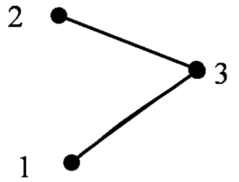

Fig. 2. *Calculation scheme: 13 – line segment of characteristic C^+; 23 – line segment of characteristic C^-*

Thus, from axis to wall, by known characteristic the following characteristic is calculated, where indexes i and $i+1$ are the order numbers of calculated characteristics. Calculation was made until the next characteristic $C^?$ came to the point «b». As initial conditions $\theta_{in}=0$ and $M_{in}=1{,}001$ were used at the initial characteristic ae. Such initial conditions are often enough applied instead of flat sound line [5]. Besides, it should be noted that for calculation of integrals trapezoid formula was used.

Functional choice. In [4] the functional which was calculated along the characteristic was used (fig. 1):

$$J=\sqrt{\int_c^b \theta^2 dl}. \qquad (5)$$

If in calculating characteristic C_{i+1}^+ avost occurred, the formula (5) was replaced by the expression:

$$J=(x_b-x_l)+\sqrt{\int_k^l \theta^2 dl}+\int_a^b \varphi(x)dx,$$

$$\varphi(x)=\begin{cases}|f(x)|, & f'(x)<0\\ 0, & f'(x)\geq 0\end{cases}, \qquad (6)$$

where the first integral was calculated along the latter calculated characteristic C_i^+, but the second one summed up the square of nozzle contour section with negative slopes. If avost occurred when calculating characteristic C_1^+, then $x_l=0$ and $\theta=0$ along the initial characteristic C_2^+. As a result of it functional (6) has the view:

$$J=x_b+\int_a^b \varphi(x)dx. \qquad (7)$$

As it is seen, the way of functional calculation (5–7), suggested in [4], is complicated enough. And, besides, replacing formula (5) by formula (6) results in the fact that functional can be non-differentiable, being continuous.

In the given work another approach is suggested. Consider two functionals

$$J_1=\sqrt{\frac{1}{L}\int_{C_i^+}\theta^2 dl},$$

$$J_2=\sqrt{\frac{1}{L}\int_{C_i^+}((\alpha-\alpha_0)/\alpha_{in})^2 dl}, \quad L=\int_{C_i^+} dl, \qquad (8)$$

that are calculated along arbitrary characteristic C_i^+. The choice of functionals (8) is conditioned by the following reasons.

Let nozzle with known M_0 satisfy the condition of problem. Then, as it is seen in fig. 3, the first functional from (8) has the two minimums $J_1=0$. To the first minimum along characteristic ae corresponds $\theta=0$ and $\alpha_{in}=\arcsin(1/M_{in})$. To the second minimum along characteristic cb corresponds $\theta=0$ and $\alpha_0=\arcsin(1/M_0)$. In this case, as it follows from the equations (1), only the second minimum is a necessary and sufficient condition to solve the problem. Therefore, it is obvious that use of the first functional does not provide uniqueness of solution.

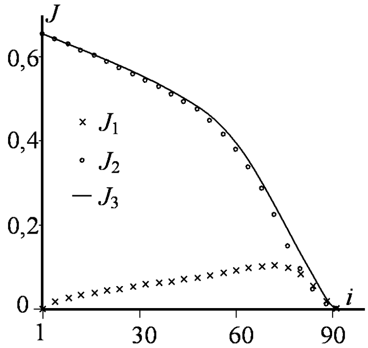

Fig. 3. *Values of functionals along characteristics C_i^+, i – characteristic number*

The second functional from (8), in contrast to the first one, has single minimum $J_2=0$ along characteristic cb (fig. 3), to which $\alpha=\alpha_0$ corresponds. However this minimum is necessary, but not sufficient condition of problem solution, since in this case $\theta=0$ can be. Combination J_1 and J_2 gives the following functional:

$$J_3=\sqrt{\frac{1}{L}\int_{C_i^+}[((\alpha-\alpha_0)/\alpha_{in})^2+\theta^2]\,dl}. \qquad (9)$$

which has single minimum $J_3=0$ along characteristic cb (fig. 3). Since this minimum is necessary anf sufficient condition of problem solution and $\theta=0$ and $\alpha=\alpha_0$ correspond to it, further functional (9) will be used.

Calculating the functional (9) let us use technique, suggested in [4], and if calculating characteristic avost occurs, in this case the functional (9) is calculated along the characteristic calculated before.

If nozzle has negative slope $f'(x_a)<0$ in point «a», avost occurs when calculating characteristic and functional (9) is calculated along the initial characteristic ae. Since in calculation of flow parameters at angular point «a» Prandtl-Mayer's formula is used (2), the one can use this formula to calculate α_2 at $\theta_2<0$. In this case the obtained values of α_2 do not have physical meaning, at subsonic flow corresponds to it. However due to the continuity (fig. 4) they permit to prolong the functional (9) onto simply connected domain in case of negative slope of nozzle $f'(x_a)<0$ and to provide continuity and differentiability for the functional (9). As the calculations show, it is enough to take only two points «a» and «e» when calculating trapezium of functional (9) by the formula at the initial characteristic ae.

As at the nozzle output boundary conditions do not change, but flow parameters are defined only by nozzle profile (4), the value of the functional (9) depends on this profile implicitly. Thus, problem of finding nozzle profile $f(x)$, доставляющего экстремум функционалу (9), сводится к поиску точки $(c_1,...,c_N)$, at which function of many variables $J_3=J_3(c_1,...,c_N)$ has extremum.

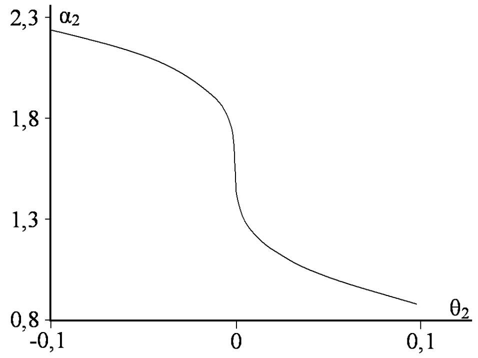

Fig. 4. *Dependence* $\alpha_2=\alpha_2(\theta_2)$

To find minimum of this function quasi-Newtonian Broiden's method is used from [6].

5. Numerical results

The number N varied variables c_i changed from 1 to 10, the initial value c_i=0. Along characteristic *ae* 50 points are given. Coordinates of points «*a*» and «*b*» are (0; 1) and (3,576; 1,299). To these values corresponds the obtained from [2] reference nozzle with angular point and Mach's number at the output M_0=2.

Comparison of solution for both functionals with reference nozzle showed that at N=10 ordinates of nozzles differed in the fourth sign after point, but maximum relative error along the ordinate amounted 0,02 %.

In fig. 5 the results of minimization are presented in the form of final values of functional for different number N, the table demonstrates the influence of coefficient number and functional used both on the number of direct calculations and the number of avosts. It is seen from the results presented that the suggested functional (9), despite its simplicity, is as good as the functional (5–7) suggested in [4] for its efficiency.

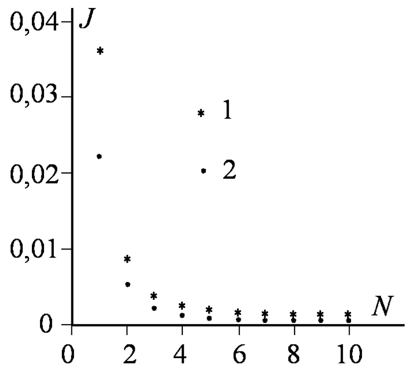

Fig. 5. *Minimization of functionals: 1) formulae (5–7) from [4], 2) formula suggested (9)*

Table. *Minimization of functionals. In numerator there is the number of direct calculations, in denominator is that of avosts*

№	Formulae (5–7) [4]	Suggested formula (9)
1	13/2	17/2
2	36/3	39/4
3	63/4	69/5
4	98/4	99/5
5	135/5	115/7
6	150/7	166/7
7	199/7	203/6
8	273/6	304/7
9	385/8	320/8
10	444/7	492/9

Conclusion

The numerical investigation performed show that suggested functional is simple enough to solve in the presence of avosts has extension on simply connected domain and keeps differentiability, being as good as the earlier considered functional in the work [4]. Comparison of profile of the nozzle obtained with reference profile shows that at 10 varied variables maximum relative error along the ordinate amounts 0,02 %.

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