New effective moduli of isotropic viscoelastic composites. Part II. Comparison of approximate calculation with the analytical solution

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Abstract. A new approximate algorithm for calculating a stress-strain state of viscoelastic
bodies is used. The algorithm is based on the derivation of the expressions of time-effective
modules. These modules are obtained by iterative changes, compressing the fork of Voigt-
Reuss. As an example the analytic solution about the action of a concentrated force on the
viscoelastic half-space is compared with the approximate solution. Numerical calculations are
performed for a wide range of relaxation characteristics of a viscoelastic half-space. Results of
the comparison of stresses and displacements with the analytic solution give coincidence
within 3…15 %.

1. Introduction

Calculations of the stress-strain state of linear viscoelastic bodies are associated with the solution of
the system of integral-differential equations of equilibrium for given stresses or displacements on the
border [1]. Approximate approaches are used in connection with the complexity of numerical
implementation of such decisions. The most famous of them are: a) the Scheper method [2], b) the
method of quasi constant operators [3], an iterative method [4], the approximation method of Il'yushin
[1]. An approximate method of the solution based on the concept of time-effective modules [5] is used
in the present work.

The purpose of this work is to carry out the calculations of the stress-strain state for a viscoelastic
body with new time-effective characteristics [6] by the example of Boussinesq's problem.

2. The action of a concentrated force on the viscoelastic half-space

Exact solutions of the problems of linear viscoelasticity can be obtained on the base of known
analytical solutions of elasticity problems. According to the method of Volterra, the elastic constants
are replaced by the corresponding integral operators. Then, decoding is performed − some operator
function is transformed to an integral equation with the unknown nucleus and the calculation of
integral is performed by a given time-function load.

We consider the elastic solution of the Boussinesq's problem [7] about the action of force P on a
half-space. We have the following axially symmetric distribution of the components (displacements
and stresses) in cylindrical coordinates (r, z):

$$u(r,z) = [\frac{r\,z}{4\pi R^3}\frac{1}{G} - \frac{r}{4\pi R(R+z)} f_1(\nu,G)]P,$$

$$w(r,z) = [\frac{1}{2\pi R} f_2(\nu,G) - \frac{z^2}{4\pi R^3}\frac{1}{G}]P,$$

$$\sigma_r(r,z) = [\frac{1}{2\pi R(R+z)} f_3(\nu,G) - \frac{3}{2\pi}\frac{r^2 z}{R^5}]P,$$ (1)

$$\sigma_\varphi(r,z) = -(\frac{R}{R+z} - \frac{z}{R})\frac{1}{2\pi R^2} f_3(\nu,G)\,P.$$

Here r, z are the radial and axial coordinates of an arbitrary point of a half- space, $R^2 = r^2 + z^2$, ν, G are resiliently instant Poisson's ratio and shear modulus.

$$f_1(\nu,G) = \frac{1-2\nu}{G}, \quad f_2(\nu,G) = \frac{1-\nu}{G}, \quad f_3(\nu,G) = (1-\nu).$$ (2)

Axial σ_z and shearing τ_{rz} stresses do not depend on the elastic constants, therefore, they will not depend on time.

We define material operators of shearing relaxation G^* and creep G^{*-1} as:

$$G^* x \equiv G(1 - \lambda \Im^*_{\lambda+\gamma})x,$$

$$G^{*-1} x \equiv \frac{1}{G}(1 + \lambda \Im^*_\gamma)x,$$ (3)

$$\Im^*_\gamma x \equiv \int_0^t e^{-\gamma(t-\tau)} x(\tau) d\tau.$$

Here γ, λ are the function parameters of relaxation and creep.

We need to make substitutions $\nu \to \nu^*, G \to G^*$ in (11) for the construction of an analytical solution of the problem and then we have to decipher the function of operators ν^*, G^*. We assume that the volumetric relaxation is absent, $K^* = K_0 = const$

We introduce the following operators:

$$\omega^* = \frac{2G^*}{3K_0}, \quad g^*_{1/2} = \frac{1}{1 + \frac{1}{2}\omega^*}.$$

Here K_0 is the bulk modulus. Then operator ν^* is expressed by the operator of related creep $g^*_{1/2}$ as follows: $\nu^* = \frac{1}{2} g^*_{1/2}(1 - \omega^*)$. The product of two integral operators of Volterra type enters the last relation. We use the transformation of the product of two operators into their difference by formula

$$\frac{1}{2}\omega^* g^*_{1/2} = 1 - g^*_{1/2}.$$ (4)

Then we obtain:

$$f_1(\nu^*,G^*) = \frac{1}{K_0} g^*_{1/2}, \quad f_2(\nu^*,G^*) = \frac{1}{2}(G^{*-1} + \frac{1}{K_0} g^*_{1/2}), \quad f_3(\nu^*,G^*) = 2 - \frac{3}{2} g^*_{1/2}.$$ (5)

As one can see, all three operator functions are expressed in terms of the operator of related creep by Il'yushin.

$$g_{1/2}^* x = \frac{2}{2+\omega_0}[1+\lambda\mu \mathfrak{I}_q^*]x,$$

$$q = \lambda + \gamma - \lambda\mu, \quad \mu = \frac{\omega_0}{2+\omega_0}, \quad \omega_0 = \frac{2G}{3K_0}. \tag{6}$$

Thus, formulas (5) in conjunction with (3) and (4) give the analytic solution for the problem on loading of the viscoelastic half-space by force $P(t)$.

We obtain approximate solutions on base $G_V(t)$, $G_R(t)$, $G_*(t)$ by replacements:

$$f_1^k = \frac{3}{3K_0 + g_k(t)}, \quad f_2^k = \frac{1}{2}(\frac{1}{g_k(t)} + \frac{3}{3K_0 + g_k(t)}), \quad f_3^k = 2 - \frac{3}{2}\frac{3K}{3K_0 + g_k(t)}, \tag{7}$$

where the index accepts values $k = 1, 2, 3$.

Here $G_V(t) = \gamma g_C(t) + (1-\gamma)g_L(t)$, $G_R(t) = \left(\frac{\gamma}{g_C(t)} + \frac{1-\gamma}{g_L(t)}\right)^{-1}$,

$$g_1(t) = g_L(t), \quad g_2(t) = g_C(t), \quad g_3(t) = G_*(t) = \sqrt{G_V(t)G_R(t)}, \tag{8}$$

where $g_L(t)$, $g_C(t)$ are known modules of Lagrange and Castigliano.

2. The analysis of numerical results

Numerical calculations of analytical and approximate solutions for Boussinesq's problem were made for three cases, where parameter η that determines the ratio of the resiliently instant module to long-termed module $\eta = G_0 / G_\infty$, takes the values equal to 3, 5, 10.

1) $\lambda = 0.0138 \text{ min}^{-1}$, $\gamma = 0.0069 \text{ min}^{-1}$, $\eta = 3$,

2) $\lambda = 0.0276 \text{ min}^{-1}$, $\gamma = 0.0069 \text{ min}^{-1}$, $\eta = 5$,

3) $\lambda = 0.0621 \text{ min}^{-1}$, $\gamma = 0.0069 \text{ min}^{-1}$, $\eta = 10$.

The boundary load was set as step function $P(t) = P_0 h(t)$, where $P_0 = 100$ MPa. The values of the elastic constants are $G_0 = 120$ MPa, $K_0 = 360$ MPa. The volume content of one component which properties are determined by time-effective characteristics of Castigliano's type, has magnitude $\gamma = 0.8$. Curves of changes in time-effective modules of Voigt $G_V(t)$, Reuss $G_R(t)$ and $g_k(t)$, $(k = 1, 2, 3)$ are shown in Figure 1.
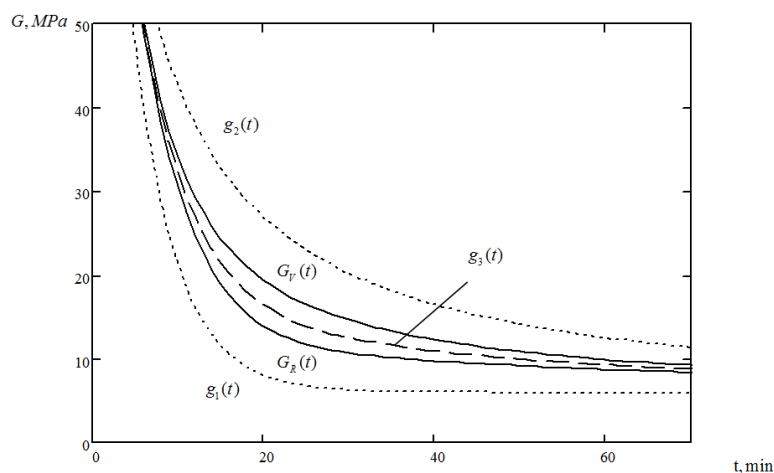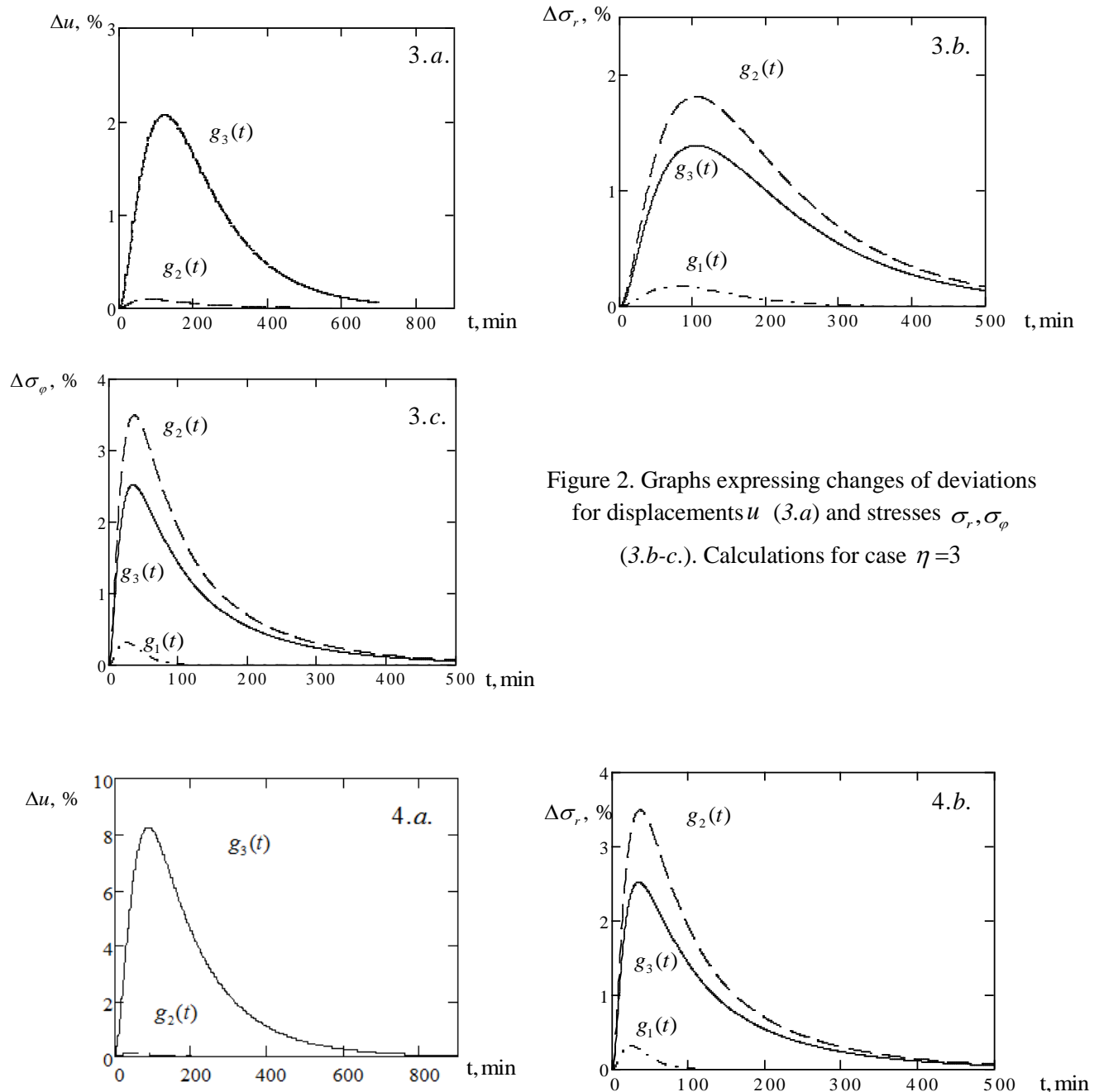

Figure 1. Curves of values for time-effective modules. $G_V(t)$, $G_R(t)$ are evaluations of Voigt and Reis, $g_1(t)$, $g_2(t)$, $g_3(t)$ are modules defined by (8), $\eta = 20$, $\gamma = 0.8$.

The curves of the time variation of the relative deviations of displacements u and stresses σ_r, σ_φ, obtained on the basis of efficient modules $g_n(t), n = 1, 2, 3$, are shown in Figures 2-4. Figures 2-4 illustrate the temporal character and the deviations of the approximate solutions against analytical solutions for the chosen values $\eta = 3, 5, 10$.
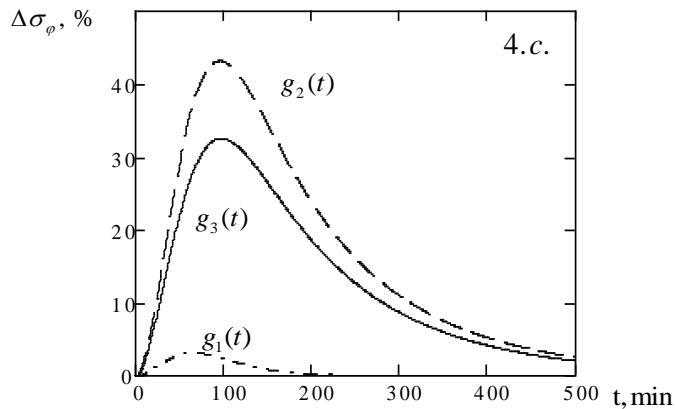

Figure 2. Graphs expressing changes of deviations for displacements u (*3.a*) and stresses σ_r, σ_φ (*3.b-c.*). Calculations for case $\eta = 3$

Figure 3. Graphs expressing changes of deviations for displacements u (*4.a*) and stresses σ_r, σ_φ (*4.b-c.*). Calculations for case $\eta = 5$
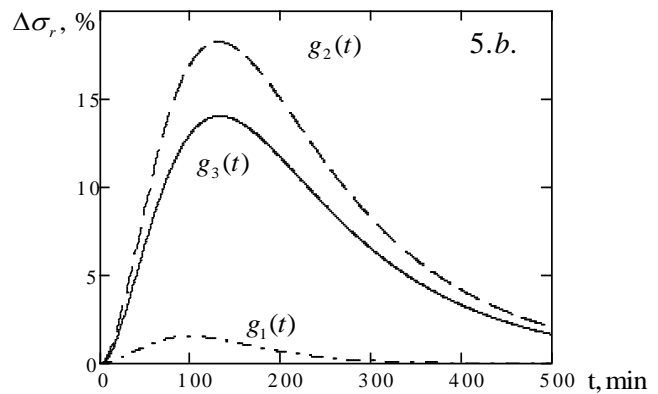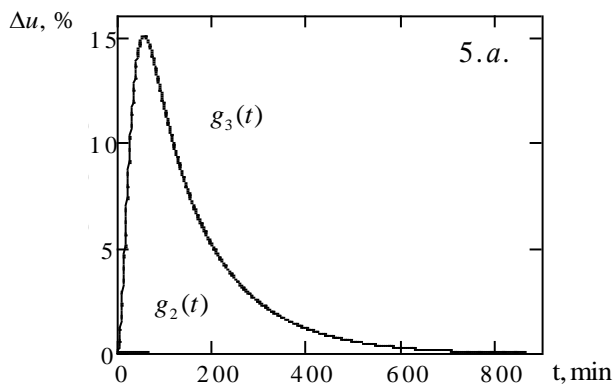

Figure 4. Graphs expressing changes of deviations for displacements u (*5.a*) and stresses σ_r (*5.b.*). Calculations for case $\eta = 10$.

3. Discussing the results

Calculations of the stress-strain state of a viscoelastic half-space (Boussinesq problem) with the help of the approximate method have a feature. It is an error in the calculation of stresses and displacements of different magnitudes. So, for a step-by-step load, the error of the calculation for displacements is practically zero when using time-effective modules of Castigliano's type. An opposite pattern is observed in the calculation of the stress: in this case, the calculations with effective modules of Lagrange's type give the minimum error. In this connection, we can use mathematical techniques applied in the mechanics of composite materials: a selection of the optimal value for the specific volume content of a component from a position of the minimum error for calculations of stresses and displacements; administering an effective time module obtained by compressing the fork of Voigt-Reuss.

4. Conclusions

1. The comparison of analytical and approximate solutions gives the following picture for the distribution of errors. The calculations of stresses and displacements give the minimal error (2...3 %) under $\eta = 3$. The deviations for displacements are less than 8 % under $\eta = 5$. Under $\eta = 10$ deviations for stresses and displacements are less than 15 %.

2. Value $\gamma = 0.8$ corresponds to the minimum magnitude of the error in the calculation of stresses and displacements.

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