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Modelling the Spectrum of Potency a Stationary Random Process in the Form of Spline First Order at the Random Number of Data in Instant of Time

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Abstract. The spectrum of potency as same the function of correlation is one of the most important characteristic of the second process random order. The spectrum of potency allows to judge about, that structure of process gives opportunity to take estimation of spectrum composition of useful signals and hindrances, allows to produce synthesis (reconstruction) the signals and to build filters and to obtain the estimates filtration. The purpose of this presenting work is modeling of process random stationary potency spectrum by random dates number in measurement moments. Using by probability theory methods and mathematical statistics was derived unbiased estimator of the potency spectrum in the form of spline first order, and were researched statistic characteristics estimation.

1. Introduction

One of the modern methods digital processing of signals is digital the spectrum analysis. It is based on modeling and studying the spectrum potency of signals, which is accepted as a some random process [1]. Density function the spectrum of potency defines the distribution of the dispersion a random process the frequency [2]. There are three basic ways of obtaining function the density of the spectrum:

1. Determination density of the spectrum $S(\omega)$ a given correlation function $R[\tau]$. Computation $S(\omega)=\frac{1}{2\pi}\int_0^{\infty}R[\tau]\cos(\omega\tau)d\tau$ difficulty, because this integral not always computed in elementary function, and the correlation function known, as rule, on the discrete set of the variable τ [3−7].
2. Estimation the density of the spectrum with the help of procedure using fast Fourier transforms. Such approach for at the spectrum analysis virtual and, as rule, to provide deriving acceptable results. However, this approach has a meaningful constraint, to wit, limiting the frequency resolution, i.e. ability to distinguish the frequency lines two or more signals [4].

3. To classic methods the spectrum of potency modeling applies periodogram methods, which in the discrete Fourier transformation conform to direct for succession, acquired realization quantization of a random process. However, as shows in [8], periodogram is not consistent estimation the spectrum of potency.
4. In the works [9-11] set, that there is a relation the spectrum of potency with a fractal characteristics a random process. Aforementioned a relation expressed as following: $S(\omega)=\frac{c}{\omega^{5-2D}}$, where c – the scaling constant, dependent on the amplitude of the signal, D – the fractal dimension, moreover this intercommunication carry estimating disposition, because she is asymptotically solution of integral equation. [12-15].
5. If about the investigated process notorious some plurality of the information, allowing pick acceptable approximation of the process, in this occurrence can get more precise the spectrum estimation, previously defined axes, chosen the model by result dimension [5].

The aim of this work is modeling the spectrum of potency a random fixed process in the form of spline first order at random number data at the time of determined as follows [16-18]. The function $S(t)$, is defined and continuous on the interval $[a, b]$, is denominated a polynomial spline of order m with clusters $x_j \in \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}$,if on each of the interval $[x_{j-1}, x_j)$, $j=\overline{1,n}$ $S(t)$ there is an algebraic polynomial of power m.

2. Statement of problem

Let the values of the process $y(t)$ are measured on the interval at time $[0;T]$. The moments of measure t_i, $i=\overline{1,N}$ are knows precisely. We will assume, that $y_i = y(t_i)+\xi_i$, $\xi(t)$ – the random function, the presence of which is caused by measurement errors, external hindrance etc. Supposed, that hindrance measurement $\xi_i=\xi(t_i)$, i=1, 2, … – independent, identically distributed a random expressions with mathematical waiting $M[\xi_i]=0$ and dispersion $D[\xi_i]=\sigma^2$. We knows the sequence of values $y_1, y_2, ..., y_N$, where $y_i=\frac{y_{i1}+y_{i2}+...+y_{in_i}}{n_i}$. Here n_i – independent a random expression, distributed by the Poisson law with argument λ and $M[y(t_i)]=0$, $M[y(t_j)y(t_i)]=R[t_j-t_i]$. By results of observations required to construct estimation $\hat{S}(\omega)$ the spectrum of potency $S(\omega)$ in the form of spline first order. A similar problem is considered in the work [19], however, it used a different approach to obtaining a spline, to wit, approach in which first searched estimation the correlation function, then, is estimation the spectrum of potency, moreover coefficients of the spline are estimated everything at once.

3. Solution of the problem

We divide the whole frequency axis ω on the interval $[0;\Omega]$, $[\Omega;2\Omega]$, $[2\Omega;3\Omega]$, …. We will consider the statistics,

$$Q=\frac{1}{\lambda^2\pi T}\sum_{i,j;\,i\neq j} n_i n_j y_i y_j \varphi(t_j-t_i), \tag{1}$$

where $\varphi(\tau)$ for the function evenly condition:

$$\varphi(\tau)=\int_{-\infty}^{+\infty}\Phi(\omega)\cos\omega\tau d\omega, \tag{2}$$

which in the integral $\int_{-\infty}^{+\infty}\Phi(\omega)\cos\omega\tau d\omega$ converges. By averaging (1) by the expression n_i, we obtain the expression $\overline{\mathrm{Q}}=\frac{1}{\pi T}\sum_{i,j;\,i\neq j} y_i y_j \varphi(t_j-t_i)$.

We seeing, that $M[\overline{Q}]=\frac{1}{\pi T}\sum_{i,j;\,i\neq j} R[t_j-t_i]\varphi[t_j-t_i]$. May assume, that the following holds approximation: $M[\overline{Q}]\approx\frac{1}{\pi T}\int_0^T\int_0^T R[u-v]\varphi(u-v)dudv$. After the change of variable, we will get $\int_0^T\int_0^T R[u-v]\varphi(u-v)dudv=\begin{vmatrix}u-v=z\\ du=dz\end{vmatrix}=\int_0^T dv\int_{-v}^{T-v}R(z)\varphi(z)dz$. We interchange the order of integration, by using the range of integration ,on the figure1.

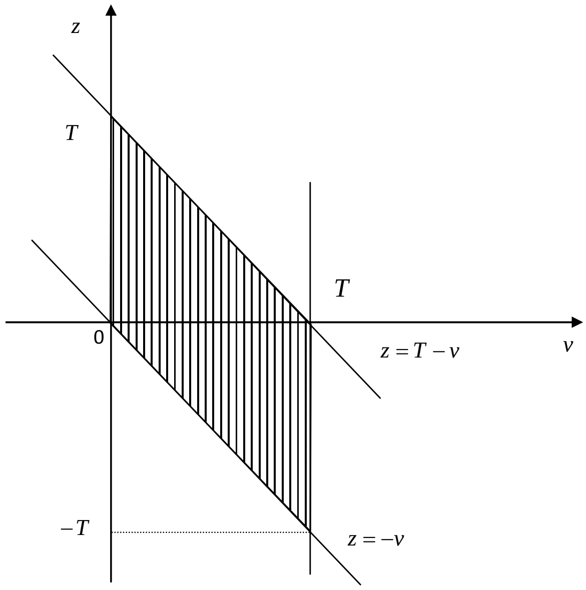

Figure 1. The range of integration.

This will allow from the original double integral go to the single integral

$$\int_{-T}^{0}R[z]\varphi(z)dz\int_{-z}^{T}dv+\int_0^T R[z]\varphi(z)dz\int_0^{T-z}dv=\int_{-T}^{0}(T+z)R[z]\varphi(z)dz+\int_0^T(T-z)R[z]\varphi(z)dz=$$

$=T\int_{-T}^{T}\left(1-\frac{|z|}{T}\right)R[z]\varphi(z)dz$. In the asymptotic case, when $T\to\infty$, the expression $\frac{|z|}{T}\to 0$ and, accordingly, $\int_{-T}^{T}\left(1-\frac{|z|}{T}\right)R[z]\varphi(z)dz\approx\int_{-\infty}^{\infty}R[z]\varphi(z)dz$. Thus

$M[\overline{Q}]\approx\frac{T}{\pi T}\int_{-\infty}^{+\infty}R[z]\varphi(z)dz=\frac{1}{\pi}\int_{-\infty}^{+\infty}R[z]\varphi(z)dz$. By using the expression (2), we will get

$M\left[\overline{Q}\right]\approx\frac{1}{\pi}\int\limits_{-\infty}^{+\infty}R[\tau]\int\limits_{-\infty}^{+\infty}\Phi(\omega)\cos\omega\tau d\omega d\tau=\frac{2}{\pi}\int\limits_{-\infty}^{+\infty}\Phi(\omega)d\omega\int\limits_{0}^{+\infty}R[\tau]\cos\omega\tau d\tau$. By using knows the ratio of Wiener – Kninchin [1], we will get

$$M\left[\overline{Q}\right]\approx\int\limits_{-\infty}^{+\infty}\Phi(\omega)S(\omega)d\omega. \tag{3}$$

For construct estimation $\hat{S}(\omega)$ the spectrum of potency $S(\omega)$ in the form of spline first order, we will assume, that on the interval $[(k-1)\Omega\,;k\Omega]$ the function $S(\omega)$ presented in the form $S(\omega)=S_{k-1}\frac{k\Omega-\omega}{\Omega}+S_k\frac{\omega-(k-1)\Omega}{\Omega}$. Consider the integral $I=\int\limits_{(k-1)\Omega}^{k\Omega}S(\omega)\left(A\frac{k\Omega-\omega}{\Omega}+B\frac{\omega-(k-1)\Omega}{\Omega}\right)d\omega$, computation of this integral leads to the expression $\frac{\Omega}{6}\left\{S_{k-1}(2A+B)+S_k(A+2B)\right\}$. Noticing that when $A=2$ и $B=-1$ $I=\frac{\Omega}{2}S_{k-1}$. Then

$$\Phi(\omega)=\begin{cases}\frac{2}{\Omega}\left(2\frac{k\Omega-\omega}{\Omega}-\frac{\omega-(k-1)\Omega}{\Omega}\right), & \omega\in\left[(k-1)\Omega;k\Omega\right],\\ 0, & \omega\notin\left[(k-1)\Omega;k\Omega\right].\end{cases}$$

By using the expression (2), we find $\varphi(\tau)=\int\limits_{(k-1)\Omega}^{k\Omega}\frac{2}{\Omega}\left(3k-1-3\frac{\omega}{\Omega}\right)\cos\omega\tau d\omega=$

$$=\begin{cases}\frac{2}{\Omega}\left\{\frac{3}{\Omega\tau^2}\left(\cos(k-1)\Omega\tau-\cos k\Omega\tau\right)-\frac{\sin k\Omega\tau}{\tau}-\frac{2\sin(k-1)\Omega\tau}{\tau}\right\}, & \omega\in\left[(k-1)\Omega,k\Omega\right],\\ 0, & \omega\notin\left[(k-1)\Omega,k\Omega\right].\end{cases}$$

We assume that $k=1$ $\varphi(\tau)=\varphi_0(\tau)$, at $k=2$ $\varphi(\tau)=\varphi_1(\tau),\ldots$. Substituting into expression (1) found of values $\varphi(\tau)$, we find the consistently $S_0, S_1,\ldots$, which we join by straight line segments, that gives on estimate $S(\omega)$ in the form of spline first order. From constructing estimates of the coefficients spline it follows, that the resulting estimates of knots of the spline are unbiased.

4. The estimate of the dispersion obtained estimates the spectrum of potency

Now we find an asymptotic estimate of the dispersion $D[Q]$ the statistic Q at $T\to\infty$. For this consider $Q^2=\frac{1}{\lambda^4\pi^2T^2}\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{k=1}^{N}n_jn_in_ln_ky_jy_iy_ly_k\varphi(t_j-t_i)\varphi(t_l-t_k)$. By averaging of this expression by values n_i, we will get $\overline{Q}^2=\frac{1}{\pi^2T^2}\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{k=1}^{N}y_jy_iy_ly_k\varphi(t_j-t_i)\varphi(t_l-t_k)$. By averaging by realization of the process $y(t)$ at fixed moments of measurement, we will get

$$M\left[y_jy_iy_ly_k\right]=M\left[y(t_j)y(t_i)y(t_l)y(t_k)\right]=R\left[t_j-t_i\right]R\left[t_l-t_k\right]+R\left[t_j-t_l\right]R\left[t_i-t_k\right]+$$
$$+R\left[t_j-t_k\right]R\left[t_i-t_l\right].$$

When

$$M\left[\overline{Q}^2\right]=\frac{1}{\pi^2T^2}\sum_{j=1}^{N}\sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{k=1}^{N}\varphi(t_j-t_i)\varphi(t_l-t_k)\left(R\left[t_j-t_i\right]R\left[t_l-t_k\right]+R\left[t_j-t_l\right]R\left[t_i-t_k\right]+\right.$$
$$\left.+R\left[t_j-t_k\right]R\left[t_i-t_l\right]\right) \quad (4)$$

Next to consider by averaging over the moments of measurement each term contains a factor of the form $R\left[t_j-t_i\right]R\left[t_l-t_k\right]$ expressions (4). Here possible the following variants:

1. All index j, i, l, k various. Making the change t_j on u, t_i on u', t_l on v, t_k on v' we get $\frac{1}{\pi^2T^2}\int_0^T\int_0^T\int_0^T\int_0^T R[u-u']R[v-v']\varphi(u-u')\varphi(v-v')dudvdu'dv'$, the computation which gives as $T\to\infty$ the square of the expectation the statistic Q, to wit, $\frac{1}{\pi^2}\left(\int_{-\infty}^{+\infty}R[z]\varphi(z)dz\right)^2$, which is subtracted when finding the dispersion.

2. Two indexes are equal, to wit, $i=k$, i.e. $t_i=t_k$, when considering summand of the expression(4) will have the form $\frac{1}{\pi^2T^2}\int_0^T\int_0^T\int_0^T R[u-u']R[v-u']\varphi(u-u')\varphi(v-v')dudvdu'$, which after the change of variables and changes the order of integration may be reduced to the form $\frac{2}{\pi^2T^2}\left\{T\int_0^T R[z]\varphi(z)dz\int_{z-T}^{T}R[t]\varphi(t)dt+\int_0^T R[z]\varphi(z)dz\int_{z-T}^{T-z}tR[t]\varphi(t)dt\right\}$, which shows, that attached to $T\to\infty$ the main contribution to the dispersion will add summand $\frac{1}{\pi^2T}\left(\int_{-\infty}^{+\infty}R[z]\varphi(z)dz\right)^2$. In case equality of indices: j and k, i and l, j and l.

3. Two pairs of indexes are equal, to wit, $j=l$, $i=k$, when considering summand of the expression (4) will have the form $\frac{1}{\pi^2T^2}\int_0^T\int_0^T R[u-u']R[u-u']\varphi(u-u')\varphi(u-u')dudu'$. Making the substitution and by changing the order of integration this the integral reduced to the form $\frac{1}{\pi^2T^2}T\int_{-T}^{T}\left(1-\frac{|z|}{T}\right)R^2[z]\varphi^2(z)dz$, in which $T\to\infty$ seeks to $\frac{1}{\pi^2T}\int_{-\infty}^{+\infty}R^2[z]\varphi^2(z)dz$.

Having considered also the other two expressions (4), after the conducted research it can be argued, that the dispersion statistic attracted to Q at $T\to\infty$ decreases as $\frac{1}{T}$.

5. The simulation modeling obtained estimated

All calculation and construction of the graph performed in the program Mathcad 2000. Now we present some results the simulation modeling obtained estimated with a shot discussion of the results. Below presented the result of modeling realization a stationary random process, which described by difference equation the first order

$$x(n)=0.8x(n-1)+\nu(n), \quad (6)$$

where $v(j) = \frac{1}{j}\sum_{i=1}^{j}\xi_i$, the values j - a random values , distribution by the Poisson law with arguments $\lambda = 3$, ξ_i - normal distribution a random values with $M[\xi_i] = 0$ and $D[\xi_i] = \sigma^2$.

As a first element generates realization $x(0)$ was taken the value 1.5. The modeling result of the realization a stationary random process, present on the figure 2.

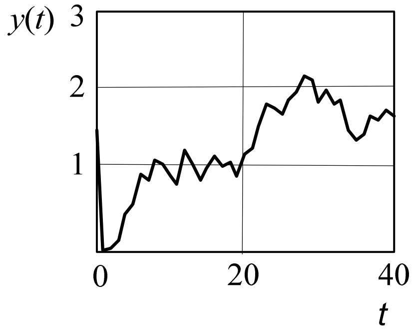

Figure 2. A model realization a stationary random process.

On the following pictures constructed estimation the spectrum of potency, the correlogram method and in the form of spline first order.

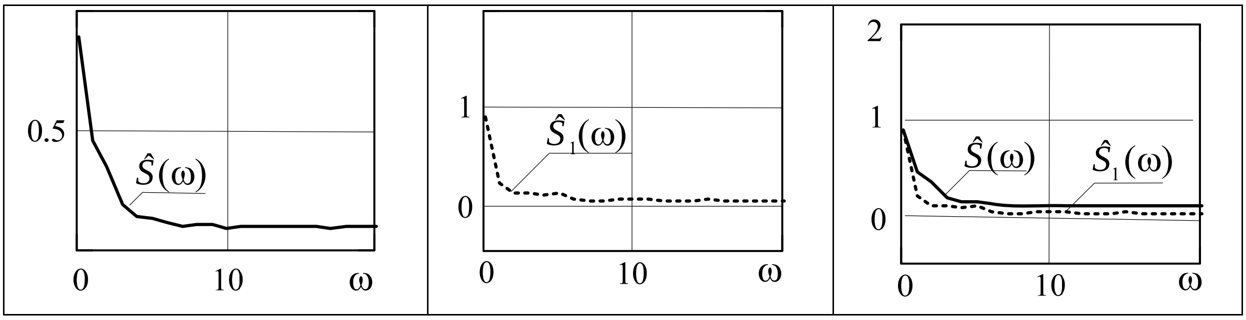

Figure 3. Presents estimated: $\hat{S}(\omega)$ the spectrum of potency $S(\omega)$, the correlogram method , $\hat{S}(\omega)$– in the form of spline first order for realization a random process figure 2

To evaluate the significance obtained estimates the spectrum of potency. For quality studies obtained a model, we use the Fisher – Snedecor. For this we will find the observed value of the criterion

$$F = \frac{\dfrac{\sum_{i=1}^{n}\left(\hat{S}_{1i} - \bar{S}\right)^2}{m-1}}{\dfrac{\sum_{i=1}^{n}\left(S_i - \hat{S}_{1i}\right)^2}{n-m}}$$

, here m –the numbers of estimated parameters, n – sample size, $\bar{S}$ – sample mean, $S_i, i = \overline{1,n}$ – values the spectrum for the model realization on a random process (6),which has the form: $S_i = \frac{0.09}{0.41 - 0.4\cos i}$ and $\hat{S}_{i1}$, $i = \overline{1,n}$ – evaluated values the spectrum of potency in the form of spline. The observed value of the criterion was equal to 23. 684, that to surpassed by far table value $F_{\alpha,m-1,n-m}$ = 3.6 the criterion Fisher-Snedecor's at the significance level $\alpha = 0.05$ and the number of degrees of freedom $k_1 = m - 1$ and $k_2 = n - m$. As $F > F_{\alpha,k_1,k_2}$, then obtained evaluation the spectrum of potency in the form of spline first order is important. The index of correlation

$$R_{\hat{S}\omega} = \sqrt{1 - \frac{\sum_{i=1}^{n}\left(S_i - \hat{S}_{1i}\right)^2}{\sum_{i=1}^{n}\left(S_i - \bar{S}\right)^2}}$$

=0.997 shows, that the estimates obtained the spectrum in the form of spline first order closely related with analytical form the spectrum a random process (6).

Thus, the spline evaluation right shows a form the spectrum of potency.

6. Conclusion

Was performed the calculation of the coefficient of spline first order in the annex to evaluation the spectrum of potency a stationary random process, when at each instant of time is a random number of measurement.

It is shows, that the obtained estimates are unbiased. It is established, that the dispersion of the estimates asymptotically behaves as $1/T$, where T − time of observation.

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