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$\mathcal{N} = 2$ supersymmetric odd-order Pais–Uhlenbeck oscillator

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Abstract

We consider an $\mathcal{N} = 2$ supersymmetric odd-order Pais–Uhlenbeck oscillator with distinct frequencies of oscillation. The technique previously developed in [Bolonek and Kosiński (2005) [7]], [Masterov (2016) [10]] is used to construct a family of Hamiltonian structures for this system.

1. Introduction

A systematic way to construct a Hamiltonian formulation for nondegenerate higher-derivative mechanical systems is based on Ostrogradsky's approach [1]. Canonical formalism for degenerate higher-derivative models can be obtained with the aid of Dirac's method for constrained systems [2] or by applying the Faddeev–Jackiw prescription [3].

However, some higher-derivative models are multi-Hamiltonian. The simplest example of such systems is the one-dimensional fourth-order Pais–Uhlenbeck (PU) oscillator [4]. Ostrogradsky's Hamiltonian of this system is unbounded from below. As a consequence, quantum theory of the model faces ghost-problem (see, e.g., a detailed discussion in Ref. [5]). For distinct frequencies of oscillation, this Hamiltonian can be presented as a difference of two harmonic

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oscillators by applying an appropriate canonical transformation [4,6]. This representation provides two functionally independent positive-definite integrals of motion. As was observed in [7] (see also Ref. [8]), a linear combination involving arbitrary nonzero coefficients of these constants of motion can also play a role of a Hamiltonian for the fourth-order PU oscillator.[1] Thus, for positive coefficients, the alternative Hamiltonian is positive-definite and consequently is more relevant for quantization than Ostrogradsky's one.

For arbitrary odd and even orders, the PU oscillator with distinct frequencies of oscillation can also be treated by the technique employed in Ref. [7]. This fact has been established in [10, 11] (see also Ref. [12]) where the corresponding families of Hamiltonian structures have been constructed. The main advantage of the alternative Hamiltonian formulation obtained in such a way is that this may correspond to a positive-definite Hamiltonian.

The even-order PU oscillator with distinct frequencies of oscillation admits an $\mathcal{N}=2$ supersymmetric extension [13]. This generalization is invariant under the time translations. However, the Noether charge associated with this symmetry can be presented as a sum of $\mathcal{N}=2$ supersymmetric harmonic oscillators which alternate in a sign [13] (see also Ref. [14]). A canonical formalism with regard to a such Hamiltonian brings about trouble with ghosts upon quantization [13]. This problem is reflected in the fact that the quantum state space of the model contains negative norm states, while a ground state is absent. In Ref. [10] an alternative Hamiltonian formulation for an $\mathcal{N}=2$ supersymmetric even-order PU oscillator has been constructed so as to avoid these nasty features.

For a particular choice of oscillation frequencies, an $\mathcal{N}=2$ supersymmetric extension of the odd-order PU oscillator has been derived in Ref. [15]. It has been shown that this extension accommodates conformal symmetry provided frequencies of oscillation form a certain arithmetic sequence. Any other aspects related with the $\mathcal{N}=2$ supersymmetric odd-order PU oscillator remain completely unexplored. In particular, a canonical formulation of this model has not yet been considered. The purpose of the present work is to construct a Hamiltonian formulation for an $\mathcal{N}=2$ supersymmetric odd-order PU oscillator with distinct frequencies of oscillation by applying the technique previously developed in Refs. [7,10].

The paper is organized as follows. In the next section we consider the odd-order PU oscillator with distinct frequencies of oscillation and introduce an $\mathcal{N}=2$ supersymmetric extension of this model. A Hamiltonian formulation for an $\mathcal{N}=2$ supersymmetric third-order PU oscillator is constructed in Sect. 3, while the general case is treated in Sect. 4. In Sect. 5, a quantum version of the $\mathcal{N}=2$ supersymmetric odd-order PU oscillator is considered. We summarize our results and discuss further possible developments in the concluding Sect. 6. Some technical details are given in Appendix. Throughout the work summation over repeated spatial indices is understood, unless otherwise is explicitly stated. Both a superscript in braces and a number of dots over spatial coordinates designate the number of derivatives with respect to time. Complex conjugation of a function f is denoted by f^*. Hermitian conjugation of an operator $\hat{a}$ is designated as $(\hat{a})^\dagger$.

2. The model

Symmetries of the PU oscillator have recently attracted some attention [16–23]. The interest was motivated by the desire to realize the so-called l-conformal Newton–Hooke algebra [24–26]

[1] An alternative Hamiltonian formulation for the fourth-order PU oscillator has been also constructed in paper [9].

in this model. As was shown in [21] (see also Ref. [27]), the $(2n+1)$-order PU oscillator, which accommodates the l-conformal Newton–Hooke symmetry, is described by the action functional[2]

$$S=\frac{1}{2}\int dt\,\epsilon_{ij}\,x_i\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+k^2\omega^2\right)\dot{x}_j, \tag{1}$$

where ϵ_{ij} is the Levi-Civitá symbol with $\epsilon_{12}=1$.

Symmetry structure intrinsic to the model (1) allows one to construct an $\mathcal{N}=2$ supersymmetric generalization of this system with the aid of Niederer-like coordinate transformations [15,35]. The action functional associated with this extension reads

$$\begin{aligned}S=\frac{1}{2}\int dt\,\epsilon_{ij}\Bigg(&x_i\prod_{k=1}^{n}\left(\frac{d^2}{dt^2}+k^2\omega^2\right)\dot{x}_j-\psi_i\left(\frac{d}{dt}+in\omega\right)\prod_{k=1}^{n-1}\left(\frac{d^2}{dt^2}+k^2\omega^2\right)\dot{\bar{\psi}}_j-\\&-\bar{\psi}_i\left(\frac{d}{dt}-in\omega\right)\prod_{k=1}^{n-1}\left(\frac{d^2}{dt^2}+k^2\omega^2\right)\dot{\psi}_j-z_i\prod_{k=1}^{n-1}\left(\frac{d^2}{dt^2}+k^2\omega^2\right)\dot{z}_j\Bigg).\end{aligned} \tag{2}$$

The configuration space of this model involves the real bosonic coordinates x_i, the fermionic coordinates ψ_i, $\bar{\psi}_i$, which are complex conjugates of each other $\bar{\psi}_i=\psi_i^*$, and real extra bosonic coordinates z_i. The model (2) is invariant under the supersymmetry transformations of the form [15]

$$\begin{aligned}&\delta x_i=i\psi_i\alpha+i\bar{\psi}_i\bar{\alpha},\qquad \delta z_i=\left(-\dot{\psi}_i+in\omega\psi_i\right)\alpha+\left(\dot{\bar{\psi}}_i+in\omega\bar{\psi}_i\right)\bar{\alpha},\\&\delta\psi_i=\left(\dot{x}_i+in\omega x_i-iz_i\right)\bar{\alpha},\qquad \delta\bar{\psi}_i=\left(\dot{x}_i-in\omega x_i+iz_i\right)\alpha,\end{aligned} \tag{3}$$

where α and $\bar{\alpha}$ are odd infinitesimal parameters.

It is evident that the model (1) can be generalized to the case of arbitrary distinct oscillation frequencies. For this purpose, the action (1) can be transformed to the form [11]

$$S=\frac{1}{2}\int dt\,\epsilon_{ij}\,x_i\prod_{k=0}^{n-1}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\dot{x}_j. \tag{4}$$

For definiteness, we assume that $0<\omega_0<\omega_1<..<\omega_{n-1}$. On the other hand, a possibility to generalize the model (2) along similar lines is less obvious, because we must simultaneously change both the action functional (2) and the supersymmetry transformations (3). By analogy with the analysis in Ref. [13], let us abandon the conformal invariance and modify the action functional (2) as follows

$$\begin{aligned}S=\frac{1}{2}\int dt\,\epsilon_{ij}\Bigg(&x_i\prod_{k=0}^{n-1}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\dot{x}_j-i\psi_i\prod_{k=-n+1}^{n-1}\left(\frac{d}{dt}+i\omega_k\right)\dot{\bar{\psi}}_j-\\&-i\bar{\psi}_i\prod_{k=-n+1}^{n-1}\left(\frac{d}{dt}-i\omega_k\right)\dot{\psi}_j-z_i\prod_{k=1}^{n-1}\left(\frac{d^2}{dt^2}+\omega_k^2\right)\dot{z}_j\Bigg),\end{aligned} \tag{5}$$

where, for convenience, we denoted $\omega_{-k}=-\omega_k$. The dynamics of this model is governed by the equations of motion

[2] Some aspects of the third-order PU oscillator have been studied in [28–30] (see also Refs. [31–34]).

$$\sum_{k=0}^{n} \sigma_k^{n,0} x_i^{(2k+1)} = 0, \ \sum_{k=0}^{2n-1} (-i)^{2n-k-1} \sigma_k^n \psi_i^{(k+1)} = 0,$$

$$\sum_{k=0}^{2n-1} i^{2n-k-1} \sigma_k^n \bar{\psi}_i^{(k+1)} = 0, \ \sum_{k=0}^{n-1} \sigma_k^{n,1} z_i^{(2k+1)} = 0,$$

where $\sigma_k^{n,s}$, σ_k^n are elementary symmetric polynomials defined by[3]

$$\sigma_k^{n,s} = \sum_{i_1 < i_2 < .. < i_{n-k} = s}^{n-1} \omega_{i_1}^2 \omega_{i_2}^2 ... \omega_{i_{n-k}}^2, \qquad \sigma_k^n = \sum_{i_1 < i_2 < .. < i_{2n-k-1} = -n+1}^{n-1} \omega_{i_1} \omega_{i_2} ... \omega_{i_{2n-k-1}}.$$

Let us show that the model (5) is an $\mathcal{N} = 2$ supersymmetric extension of the odd-order PU oscillator (4). As the first step, one finds the Noether charge which corresponds to the invariance of the model (5) under the time translations

$$\begin{aligned} H = \sum_{k=1}^{n} \sum_{m=0}^{n-k} \sigma_{k+m}^{n,0} \epsilon_{ij} x_i^{(2k)} x_j^{(2m+1)} - \sum_{k=1}^{n-1} \sum_{m=0}^{n-k-1} \sigma_{k+m}^{n,1} \epsilon_{ij} z_i^{(2k)} z_j^{(2m+1)} + \\ + (-1)^{n+1} \sum_{k=1}^{2n-1} \sum_{m=0}^{2n-k-1} i^{k-m} \sigma_{k+m}^n \epsilon_{ij} \psi_i^{(k)} \bar{\psi}_j^{(m+1)}. \end{aligned} \tag{6}$$

Dirac's Hamiltonian of the system (5) is the phase space analogue of this conserved quantity. Therefore, there exists such a graded Poisson bracket $[\cdot, \cdot\}$ that the relations

$$\begin{aligned}
&[x_i^{(k)}, H\} = x_i^{(k+1)}, \quad k = 0, 1, .., 2n-1, \quad [x_i^{(2n)}, H\} = -\sum_{k=0}^{n-1} \sigma_k^{n,0} x_i^{(2k+1)}, \\
&[\psi_i^{(k)}, H\} = \psi_i^{(k+1)}, \quad k = 0, 1, .., 2n-2, \quad [\psi_i^{(2n-1)}, H\} = -\sum_{k=0}^{2n-2} (-i)^{2n-k-1} \sigma_k^n \psi_i^{(k+1)}, \\
&[\bar{\psi}_i^{(k)}, H\} = \bar{\psi}_i^{(k+1)}, \quad k = 0, 1, .., 2n-2, \quad [\bar{\psi}_i^{(2n-1)}, H\} = -\sum_{k=0}^{2n-2} i^{2n-k-1} \sigma_k^n \bar{\psi}_i^{(k+1)}, \\
&[z_i^{(k)}, H\} = z_i^{(k+1)}, \quad k = 0, 1, .., 2n-3, \quad [z_i^{(2n-2)}, H\} = -\sum_{k=0}^{n-2} \sigma_k^{n,1} z_i^{(2k+1)},
\end{aligned} \tag{7}$$

hold. It is straightforward to verify (for some technical details see Appendix) that this bracket can be defined as follows[4]

3 By definition, we put $\sigma_{n-s}^{n,s} \equiv 1$ for $s = 0, 1$, $\sigma_{2n-1}^n \equiv 1$.

4 For the model (4), an analogue of the bracket (8) has been introduced in Ref. [11].

$$
\begin{aligned}
[A,B\} = (-1)^{n+1}\Bigg(&\sum_{r,m=0}^{2n}\mu^{n,0}_{rm}\epsilon_{ij}\frac{\partial A}{\partial x_i^{(r)}}\frac{\partial B}{\partial x_j^{(m)}}+\sum_{r,m=0}^{2n-2}\mu^{n,1}_{rm}\epsilon_{ij}\frac{\partial A}{\partial z_i^{(r)}}\frac{\partial B}{\partial z_j^{(m)}}+\\
&-\sum_{r,m=0}^{2n-1}\upsilon_{rm}\epsilon_{ij}\frac{\overleftarrow{\partial} A}{\partial \psi_i^{(r)}}\frac{\overrightarrow{\partial} B}{\partial \bar{\psi}_j^{(m)}}+\sum_{r,m=0}^{2n-1}(\upsilon_{rm})^{*}\epsilon_{ij}\frac{\overleftarrow{\partial} A}{\partial \bar{\psi}_i^{(r)}}\frac{\overrightarrow{\partial} B}{\partial \psi_j^{(m)}}\Bigg),
\end{aligned}
\tag{8}
$$

where the coefficients $\mu^{n,0}_{rm}$, $\mu^{n,1}_{rm}$, and υ_{rm} are given by

$$
\mu^{n,s}_{rm} = \begin{cases} 0 \\ (-1)^{\frac{r-m}{2}} P^{n,s}_{r+m-2n+2s} \end{cases}, \quad \upsilon_{rm} = \begin{cases} i^{r-m} P^{n,0}_{r+m-2n+1}, r+m\ -\ \text{odd} \\ \omega_0 i^{r-m} P^{n,0}_{r+m-2n}, r+m\ -\ \text{even} \end{cases},
$$

with $P^{n,s}_{2k}$ being the k-th degree symmetric polynomial in $(n-s)$ variables $\omega_s^2, \omega_{s+1}^2, .., \omega_{n-1}^2$

$$
P^{n,s}_{2k} = \sum_{\substack{\lambda_s,\lambda_{s+1},..,\lambda_{n-1}=0\\ \lambda_s+\lambda_{s+1}+..+\lambda_{n-1}=k}}^{k} \omega_s^{2\lambda_s}\omega_{s+1}^{2\lambda_{s+1}}...\omega_{n-1}^{2\lambda_{n-1}}.
$$

By definition, this polynomial is equal to zero for negative values of k. In the next sections we will show that (8) possesses the standard properties of a graded Poisson bracket.

As the next step, we need to generalize the supersymmetry transformations (3) to the case of arbitrary distinct oscillation frequencies. To this end, let us note that the transformations (3) are also available for an $\mathcal{N}=2$ supersymmetric even-order PU oscillator which exhibits conformal invariance [15]. Therefore, it is natural to expect that both the model (5) and its even-order analogue [13] are invariant with respect to the supersymmetry transformations

$$
\begin{aligned}
&\delta x_i = \psi_i\alpha + \bar{\psi}_i\bar{\alpha}, \qquad \delta z_i = \left(i\dot{\psi}_i + \omega_0\psi_i\right)\alpha + \left(-i\dot{\bar{\psi}}_i + \omega_0\bar{\psi}_i\right)\bar{\alpha},\\
&\delta\psi_i = (-i\dot{x}_i + \omega_0 x_i - z_i)\bar{\alpha}, \qquad \delta\bar{\psi}_i = (-i\dot{x}_i - \omega_0 x_i + z_i)\alpha,
\end{aligned}
\tag{9}
$$

which have been introduced in Ref. [13] for an $\mathcal{N}=2$ supersymmetric even-order PU oscillator. It is straightforward to verify that this is the case. The integrals of motion, which correspond to these transformations, read

$$
\begin{aligned}
Q = -&\sum_{k=1}^{n-1}\sum_{m=0}^{n-k-1}\sigma^{n,1}_{k+m}\epsilon_{ij}(x_i^{(2k+1)} + \omega_0^2 x_i^{(2k-1)} + i z_i^{(2k)} - \omega_0 z_i^{(2k-1)})\psi_j^{(2m+1)} +\\
&+\sum_{k=0}^{n-1}\sum_{m=0}^{n-k-1}\sigma^{n,1}_{k+m}\epsilon_{ij}(x_i^{(2k+2)} + \omega_0^2 x_i^{(2k)} - \omega_0 z_i^{(2k)})\psi_j^{(2m)} +\\
+ i&\sum_{k=0}^{n-2}\sum_{m=1}^{n-k-1}\sigma^{n,1}_{k+m}\epsilon_{ij} z_i^{(2k+1)}\psi_j^{(2m)} - \epsilon_{ij}(\dot{x}_i - i\omega_0 x_i + i z_i)\sum_{k=0}^{n-1}\sigma^{n,1}_{k}\psi_j^{(2k+1)}, \qquad \bar{Q} = Q^{*}.
\end{aligned}
\tag{10}
$$

These constants of motion, together with the Hamiltonian (6), obey the following relations

$$
\begin{aligned}
&[Q,Q\} = 0, \qquad [H,Q\} = 0, \qquad [Q,\bar{Q}\} = -2iH,\\
&[H,\bar{Q}\} = 0, \qquad [\bar{Q},\bar{Q}\} = 0,
\end{aligned}
\tag{11}
$$

with respect to the bracket (8). So, the model (5) is an $\mathcal{N}=2$ supersymmetric extension of the odd-order PU oscillator (4).

3. $\mathcal{N}=2$ supersymmetric third-order PU oscillator

According to the analysis in Ref. [11], a Hamiltonian formulation of the odd-order PU oscillator (4) is not unique. Let us generalize this result to the case of an $\mathcal{N}=2$ supersymmetric third-order PU oscillator. For $n=1$, the Hamiltonian of the model (4) can be presented as a difference of two one-dimensional harmonic oscillators [11]. This can be achieved by using the coordinates

$$q_k=\frac{1}{\sqrt{2\omega_0}}\left(\dot{x}_1+\frac{(-1)^k}{\omega_0}\ddot{x}_2\right),\quad p_k=\sqrt{\frac{\omega_0}{2}}\left(\dot{x}_2+\frac{(-1)^{k+1}}{\omega_0}\ddot{x}_1\right),$$
$$y_k=\frac{1}{\omega_0}(\ddot{x}_k+\omega_0^2x_k). \tag{12}$$

With respect to the supersymmetry transformations (9), the variables q_k and y_k are transformed as follows

$$\delta q_k=\vartheta_k\alpha+\bar{\vartheta}_k\bar{\alpha},\qquad \delta y_k=\theta_k\alpha+\bar{\theta}_k\bar{\alpha},$$

where we denoted

$$\vartheta_k=\frac{1}{\sqrt{2\omega_0}}\left(\dot{\psi}_1+i(-1)^k\dot{\psi}_2\right),\qquad \theta_k=i\dot{\psi}_k+\omega_0\psi_k,$$
$$\bar{\vartheta}_k=(\vartheta_k)^*,\qquad \bar{\theta}_k=(\theta_k)^*. \tag{13}$$

The nonvanishing structure relations between the coordinates (12), (13) read

$$[q_k,p_m\}=\delta_{km},\quad [y_k,y_m\}=-\epsilon_{km},\quad [\vartheta_k,\bar{\vartheta}_m\}=i(-1)^k\delta_{km},$$
$$[\theta_k,\bar{\theta}_m\}=\omega_0\,\epsilon_{km},\quad \text{(no sum)}.$$

Using the variables (12), (13), the Hamiltonian (6) and supercharges (10) for $n=1$ may be rewritten as[5]

$$H=\frac{1}{2}(p_1^2+\omega_0^2q_1^2+2\omega_0\vartheta_1\bar{\vartheta}_1)-\frac{1}{2}(p_2^2+\omega_0^2q_2^2+2\omega_0\vartheta_2\bar{\vartheta}_2), \tag{14}$$
$$Q=\vartheta_1(p_1-i\omega_0q_1)+\vartheta_2(p_2+i\omega_0q_2)-\epsilon_{ij}\theta_i(y_j-z_j),\qquad \bar{Q}=(Q)^*. \tag{15}$$

So, the Hamiltonian of an $\mathcal{N}=2$ supersymmetric third-order PU oscillator can be presented as a difference of two one-dimensional $\mathcal{N}=2$ supersymmetric harmonic oscillators. At first sight it may appear that an $\mathcal{N}=2$ supersymmetric odd-order PU oscillator is dynamically equivalent to a set of two decoupled $\mathcal{N}=2$ supersymmetric harmonic oscillators. This is not true because the phase spaces of these systems are not isomorphic. In addition to oscillator coordinates $(q_i,p_i,\vartheta_i,\bar{\vartheta}_i)$, the phase space of the $\mathcal{N}=2$ supersymmetric odd-order PU oscillator involves variables $a_i=\{y_i,z_i,\theta_i,\bar{\theta}_i\}$ whose dynamics are trivial $\dot{a}_i=0$. This also can be illustrated by rewriting the action functional (5) as follows (up to a total derivative term)

$$S=\frac{1}{2}\int dt\Big[\Big(\dot{q}_1^2-\omega_0^2q_1^2+i\vartheta_1\dot{\bar{\vartheta}}_1+i\bar{\vartheta}_1\dot{\vartheta}_1-2\omega_0\vartheta_1\bar{\vartheta}_1\Big)+\epsilon_{ij}\left(y_i\dot{y}_j-z_i\dot{z}_j\right)-$$
$$-\Big(\dot{q}_2^2-\omega_0^2q_2^2+i\vartheta_2\dot{\bar{\vartheta}}_2+i\bar{\vartheta}_2\dot{\vartheta}_2-2\omega_0\vartheta_2\bar{\vartheta}_2\Big)+\frac{1}{\omega_0}\epsilon_{ij}(\theta_i\dot{\bar{\theta}}_j-\bar{\theta}_i\dot{\theta}_j)\Big].$$

[5] Note that the supersymmetry algebra (11) does not change when the supercharges are redefined as follows $Q\to Q+\epsilon_{ij}\theta_i(y_j-z_j)$, $\bar{Q}\to\bar{Q}+\epsilon_{ij}\bar{\theta}_i(y_j-z_j)$.

Let us construct an alternative Hamiltonian formulation for an $\mathcal{N}=2$ supersymmetric third-order PU oscillator by applying the approach previously developed in Ref. [7]. To this end, we must deform both the Hamiltonian (14) and the corresponding Poisson bracket (8) in such a way that the equations (7) will be preserved. Let us choose the following deformation of the Hamiltonian (14)

$$\mathcal{H}=\frac{\gamma_1}{2}(p_1^2+\omega_0^2q_1^2+2\omega_0\vartheta_1\bar{\vartheta}_1)+\frac{\gamma_2}{2}(p_2^2+\omega_0^2q_2^2+2\omega_0\vartheta_2\bar{\vartheta}_2), \tag{16}$$

where γ_1 and γ_2 are arbitrary nonzero coefficients. With the change $H\to\mathcal{H}$, the equations (7) are satisfied provided the graded Poisson structure relations have the form

$$\begin{aligned}
&[x_i,\dddot{x}_j\}=-\gamma^-\epsilon_{ij}, && [x_i,\ddot{x}_j\}=\frac{1}{\omega_0}\gamma^+\delta_{ij}, && [\psi_i,\dot{\bar{\psi}}_j\}=i\gamma^-\epsilon_{ij}-\gamma^+\delta_{ij}, && [z_i,z_j\}=\epsilon_{ij},\\
&[\dot{x}_i,\ddot{x}_j\}=\gamma^-\epsilon_{ij}, && [\dot{x}_i,\dddot{x}_j\}=\omega_0\gamma^+\delta_{ij}, && [\dot{\psi}_i,\bar{\psi}_j\}=-i\gamma^-\epsilon_{ij}+\gamma^+\delta_{ij}, &&\\
&[\ddot{x}_i,\dddot{x}_j\}=\omega_0^2\gamma^-\epsilon_{ij}, && [\psi_i,\bar{\psi}_j\}=-\frac{i}{\omega_0}\gamma^+\delta_{ij}, && [\dot{\psi}_i,\dot{\bar{\psi}}_j\}=-\omega_0\gamma^-\epsilon_{ij}-i\omega_0\gamma^+\delta_{ij}, &&
\end{aligned} \tag{17}$$

where we denote

$$\gamma^\pm=\frac{1}{2}\left(\frac{1}{\gamma_1}\pm\frac{1}{\gamma_2}\right).$$

This Poisson structure is degenerate when $\gamma_1=\gamma_2$. By this reason, in what follows we exclude this case from our consideration.

Let us introduce the new variables

$$\begin{aligned}
&\mathfrak{q}_k=\sqrt{|\gamma_k|}q_k, && \mathfrak{p}_k=(-1)^{k+1}\mathrm{sign}(\gamma_k)\sqrt{|\gamma_k|}p_k, && \mathsf{y}_k=\frac{1}{\sqrt{|\gamma^-|}}y_k,\\
&\Psi_k=\sqrt{|\gamma_k|}\vartheta_k, && \bar{\Psi}_k=\sqrt{|\gamma_k|}\bar{\vartheta}_k, && \Theta_k=\frac{1}{\sqrt{|\gamma^-|}}\theta_k, \quad \bar{\Theta}_k=\frac{1}{\sqrt{|\gamma^-|}}\bar{\theta}_k.
\end{aligned} \quad \text{(no sum)} \tag{18}$$

Under the bracket (17), these coordinates obey the following nonvanishing relations

$$\begin{aligned}
&[\mathfrak{q}_k,\mathfrak{p}_m\}=\delta_{km}, && [\mathsf{y}_k,\mathsf{y}_m\}=-\mathrm{sign}(\gamma^-)\epsilon_{km},\\
&[\Psi_k,\bar{\Psi}_m\}=-i\,\mathrm{sign}(\gamma_k)\delta_{km}, && [\Theta_k,\bar{\Theta}_m\}=\omega_0\,\mathrm{sign}(\gamma^-)\epsilon_{km},
\end{aligned} \quad \text{(no sum)} \tag{19}$$

Here and in what follows $\mathrm{sign}(x)$ denotes the standard signum function. The Hamiltonian (16) in terms of the variables (18) takes the form

$$\mathcal{H}=\frac{\mathrm{sign}(\gamma_1)}{2}(\mathfrak{p}_1^2+\omega_0^2\mathfrak{q}_1^2+2\omega_0\Psi_1\bar{\Psi}_1)+\frac{\mathrm{sign}(\gamma_2)}{2}(\mathfrak{p}_2^2+\omega_0^2\mathfrak{q}_2^2+2\omega_0\Psi_2\bar{\Psi}_2). \tag{20}$$

Along with this alternative Hamiltonian, the full formulation of an $\mathcal{N}=2$ supersymmetric third-order PU oscillator involves supercharges. According to the analysis in Ref. [10], one may try to find these by using an auxiliary action functional. Taking into account the relations (19), in our case such an action can be chosen in the form

$$\begin{aligned}
\mathcal{S}=\frac{1}{2}\int dt\,&\mathrm{sign}(\gamma_1)(\dot{\mathfrak{q}}_1^2-\omega_0^2\mathfrak{q}_1^2+i\Psi_1\dot{\bar{\Psi}}_1+i\bar{\Psi}_1\dot{\Psi}_1-2\omega_0\Psi_1\bar{\Psi}_1)+\\
&+\mathrm{sign}(\gamma^-)\epsilon_{ij}\mathsf{y}_i\dot{\mathsf{y}}_j-\epsilon_{ij}z_i\dot{z}_j+\\
&+\mathrm{sign}(\gamma_2)(\dot{\mathfrak{q}}_2^2-\omega_0^2\mathfrak{q}_2^2+i\Psi_2\dot{\bar{\Psi}}_2+i\bar{\Psi}_2\dot{\Psi}_2-2\omega_0\Psi_2\bar{\Psi}_2)+\\
&+\frac{\mathrm{sign}(\gamma^-)}{\omega_0}\epsilon_{ij}\left(\Theta_i\dot{\bar{\Theta}}_j-\bar{\Theta}_i\dot{\Theta}_j\right).
\end{aligned}$$

This action is invariant under the transformations

$$\delta \mathsf{q}_k=\Psi_k\alpha+\bar\Psi_k\bar\alpha,\qquad \delta\Psi_k=(-i\dot{\mathsf{q}}_k+\omega_0\mathsf{q}_k)\bar\alpha,\quad \delta\Theta_k=\omega_0(\mathsf{y}_k-\mathrm{sign}(\gamma^-)z_k)\bar\alpha,$$
$$\delta\mathsf{y}_k=\delta z_k=\Theta_k\alpha+\bar\Theta_k\bar\alpha,\quad \delta\bar\Psi_k=(-i\dot{\mathsf{q}}_k-\omega_0\mathsf{q}_k)\alpha,\quad \delta\bar\Theta_k=-\omega_0(\mathsf{y}_k-\mathrm{sign}(\gamma^-)z_k)\alpha,$$

which yield the following Noether integrals of motion

$$\mathcal{Q}=\Psi_1(\mathfrak{p}_1-i\,\mathrm{sign}(\gamma_1)\omega_0\mathsf{q}_1)+\Psi_2(\mathfrak{p}_2-i\,\mathrm{sign}(\gamma_2)\omega_0\mathsf{q}_2)-\epsilon_{ij}\Theta_i(\mathrm{sign}(\gamma^-)\mathsf{y}_j-z_j),$$
$$\bar{\mathcal{Q}}=(\mathcal{Q})^*.$$

With respect to the alternative Poisson structure (17), these conserved quantities and the alternative Hamiltonian (20) obey the relations

$$[\mathcal{H},\mathcal{Q}\}=0,\qquad [\mathcal{Q},\bar{\mathcal{Q}}\}=-2i\mathcal{H}+(1-\mathrm{sign}(\gamma^-))\epsilon_{ij}\Theta_i\bar\Theta_j,\qquad [\mathcal{H},\bar{\mathcal{Q}}\}=0,$$
$$[\mathcal{Q},\mathcal{Q}\}=(1-\mathrm{sign}(\gamma^-))\epsilon_{ij}\Theta_i\Theta_j,\qquad [\bar{\mathcal{Q}},\bar{\mathcal{Q}}\}=(1-\mathrm{sign}(\gamma^-))\epsilon_{ij}\bar\Theta_i\bar\Theta_j,$$

Thus, for positive γ^-, we have an appropriate supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$. Moreover, if we put $0<\gamma_1<\gamma_2$ then the corresponding alternative Hamiltonian becomes a direct sum of two one-dimensional $\mathcal{N}=2$ supersymmetric harmonic oscillators.

4. The general case

Let us consider an $\mathcal{N}=2$ supersymmetric PU oscillator of arbitrary odd order. To construct an alternative Hamiltonian formulation for this system, one should obtain a more appropriate representation for the Hamiltonian (6). According to the analysis in Ref. [11], a Hamiltonian of the model (4) can be represented as a direct sum of the third-order PU oscillators which alternate in a sign. This can be achieved with the aid of the so-called oscillator variables [4,11]

$$\tilde{x}_{k,i}=\sqrt{\rho_k^{n,0}}\prod_{\substack{m=0\\m\neq k}}^{n-1}\left(\frac{d^2}{dt^2}+\omega_m^2\right)\dot{x}_i,\qquad z_{0,i}=\frac{1}{\prod_{s=0}^{n-1}\omega_s}\prod_{m=0}^{n-1}\left(\frac{d^2}{dt^2}+\omega_m^2\right)x_i,\tag{21}$$

where $k=0,1,..,n-1$; the coefficients $\rho_k^{n,s}$ are given by

$$\rho_k^{n,s}=\frac{(-1)^{k+s}}{\prod_{\substack{m=s\\m\neq k}}^{n-1}(\omega_m^2-\omega_k^2)},\qquad k=s,s+1,..,n-1.$$

Taking into account the results of Refs. [10,13], let us introduce similar variables for the remaining coordinates

$$\psi_{p,i}=\sqrt{\rho_p^n}\prod_{\substack{m=-n+1\\m\neq p}}^{n-1}\left(\frac{d}{dt}-i\omega_m\right)\dot\psi_i,$$

$$\theta_i=\frac{i}{\prod_{s=1}^{n-1}\omega_s}\prod_{m=-n+1}^{n-1}\left(\frac{d}{dt}-i\omega_m\right)\psi_i,\qquad \bar\psi_{p,i}=(\psi_{p,i})^*,$$

$$\tilde{x}_{-k,i} = \sqrt{\rho_k^{n,1}} \prod_{\substack{m=1\\ m\neq k}}^{n-1} \left(\frac{d^2}{dt^2} + \omega_m^2\right) \dot{z}_i, \tag{22}$$

$$z_{1,i} = \frac{1}{\prod\limits_{s=1}^{n-1} \omega_s} \prod_{m=1}^{n-1} \left(\frac{d^2}{dt^2} + \omega_m^2\right) z_i, \qquad \bar{\theta}_i = (\theta_i)^*,$$

where $k = 1, 2, .., n-1$, $p = -n+1, -n+2, .., n-1$; the coefficients ρ_p^n are defined by

$$\rho_p^n = \frac{(-1)^{n+p-1}}{\prod\limits_{\substack{m=-n+1\\ m\neq p}}^{n-1} (\omega_m - \omega_p)} = \frac{\omega_p + \omega_0}{2\omega_p} \rho_{|p|}^{n,0}.$$

Let us draw our attention to how the variables $\tilde{x}_{\pm k,i}$, $\psi_{\pm k,i}$, and $\bar{\psi}_{\pm k,i}$ $(k = 1, 2, .., n-1)$ are transformed under the supersymmetry transformations (9)

$$\begin{aligned}
&\delta\tilde{x}_{\pm k,i} = (\mu_k^{\pm}\psi_{k,i} \pm \mu_k^{\mp}\psi_{-k,i})\alpha + (\mu_k^{\pm}\bar{\psi}_{k,i} \pm \mu_k^{\mp}\bar{\psi}_{-k,i})\bar{\alpha},\\
&\delta\psi_{\pm k,i} = (-\mu_k^{\pm}(i\dot{\tilde{x}}_{k,i} \mp \omega_k\tilde{x}_{k,i}) \mp \mu_k^{\mp}(i\dot{\tilde{x}}_{-k,i} \mp \omega_k\tilde{x}_{-k,i}))\bar{\alpha}, \quad \text{with} \quad \mu_k^{\pm} = \sqrt{\frac{\omega_k \pm \omega_0}{2\omega_k}}.\\
&\delta\bar{\psi}_{\pm k,i} = (-\mu_k^{\pm}(i\dot{\tilde{x}}_{k,i} \pm \omega_k\tilde{x}_{k,i}) \mp \mu_k^{\mp}(i\dot{\tilde{x}}_{-k,i} \pm \omega_k\tilde{x}_{-k,i}))\alpha,
\end{aligned} \tag{23}$$

This motivates us to perform one more change of the bosonic coordinates

$$x_{-k,i} = \mu_k^{-}\tilde{x}_{k,i} - \mu_k^{+}\tilde{x}_{-k,i}, \quad x_{0,i} = \tilde{x}_{0,i}, \quad x_{k,i} = \mu_k^{+}\tilde{x}_{k,i} + \mu_k^{-}\tilde{x}_{-k,i}. \tag{24}$$

The supersymmetry transformations (23) then become

$$\begin{aligned}
&\delta x_{\pm k,i} = \psi_{\pm k,i}\alpha + \bar{\psi}_{\pm k,i}\bar{\alpha}, \quad \delta\psi_{\pm k,i} = -(i\dot{x}_{\pm k,i} \mp \omega_k x_{\pm k,i})\bar{\alpha},\\
&\delta\bar{\psi}_{\pm k,i} = -(i\dot{x}_{\pm k,i} \pm \omega_k x_{\pm k,i})\alpha.
\end{aligned}$$

The Hamiltonian (6) and the supercharges (10) in terms of $x_{k,i}$, $\psi_{k,i}$, $\bar{\psi}_{k,i}$, and $z_{s,i}$ may be represented as follows

$$H = \sum_{k=-n+1}^{n-1} (-1)^{k+1}\epsilon_{ij}(x_{k,i}\dot{x}_{k,j} - i\psi_{k,i}\bar{\psi}_{k,j}),$$

$$Q = \sum_{k=-n+1}^{n-1} \frac{(-1)^k}{\omega_k}\epsilon_{ij}\psi_{k,i}(i\dot{x}_{k,j} + \omega_k x_{k,j}) - \epsilon_{ij}\theta_i(z_{0,j} - z_{1,j}), \quad \bar{Q} = (Q)^*.$$

So, we have shown that the Hamiltonian of an $\mathcal{N} = 2$ supersymmetric $(2n+1)$-order PU oscillator can be presented as a direct sum of $(2n-1)$ $\mathcal{N} = 2$ supersymmetric third-order PU oscillators which alternate in a sign. This fact correlates with the analysis in Ref. [11] for the model (4).

By analogy with (12), (13), let us introduce the coordinates

$$q_{k,s} = \frac{1}{\sqrt{|2\omega_k|}}\left(x_{k,1} + \frac{(-1)^s}{|\omega_k|}\dot{x}_{k,2}\right), \qquad p_{k,s} = (-1)^k\sqrt{\frac{|\omega_k|}{2}}\left(x_{k,2} + \frac{(-1)^{s+1}}{|\omega_k|}\dot{x}_{k,1}\right),$$
$$\vartheta_{k,s} = \frac{1}{\sqrt{|2\omega_k|}}(\psi_{k,1} + i(-1)^s\,\mathrm{sign}(\omega_k)\psi_{k,2}), \tag{25}$$
$$\bar{\vartheta}_{k,s} = \frac{1}{\sqrt{|2\omega_k|}}(\bar{\psi}_{k,1} - i(-1)^s\,\mathrm{sign}(\omega_k)\bar{\psi}_{k,2}).$$

Given the bracket (8), these variables obey

$$[q_{k,s}, p_{m,j}\} = \delta_{km}\delta_{sj}, \qquad [\vartheta_{k,s}, \bar{\vartheta}_{m,j}\} = i(-1)^{k+s}\delta_{km}\delta_{sj}. \qquad \text{(no sum)}$$

The existence of these coordinates automatically implies that (8) possesses standard properties of a graded Poisson bracket.

In terms of the variables (25), the Hamiltonian (6) takes the form

$$H = \sum_{k=-n+1}^{n-1}(-1)^k\left[\left(\frac{1}{2}p_{k,1}^2 + \frac{\omega_k^2}{2}q_{k,1}^2 + \omega_k\vartheta_{k,1}\bar{\vartheta}_{k,1}\right) - \left(\frac{1}{2}p_{k,2}^2 + \frac{\omega_k^2}{2}q_{k,2}^2 + \omega_k\vartheta_{k,2}\bar{\vartheta}_{k,2}\right)\right].$$

Let us consider the following deformation of this Hamiltonian

$$\mathcal{H} = \sum_{k=-n+1}^{n-1}\gamma_{|k|,1}\left(\frac{1}{2}p_{k,1}^2 + \frac{\omega_k^2}{2}q_{k,1}^2 + \omega_k\vartheta_{k,1}\bar{\vartheta}_{k,1}\right)$$
$$+ \gamma_{|k|,2}\left(\frac{1}{2}p_{k,2}^2 + \frac{\omega_k^2}{2}q_{k,2}^2 + \omega_k\vartheta_{k,2}\bar{\vartheta}_{k,2}\right), \tag{26}$$

where $\gamma_{0,1}, \gamma_{0,2}, \gamma_{1,1}, .., \gamma_{n-1,2}$ are arbitrary nonzero coefficients. It is straightforward to verify (for technical details see Appendix) that the equations (7), where $H \to \mathcal{H}$, are satisfied provided the following graded Poisson structure

	$[x_i^{(s)}, x_j^{(m)}\}$	$[z_i^{(s)}, z_j^{(m)}\}$
$s = m = 0$	0	0
$s + m - odd$	$(-1)^{\frac{s-m+1}{2}}\sum_{k=0}^{n-1}\rho_k^{n,0}\omega_k^{s+m-2}\gamma_k^{+}\delta_{ij}$	$(-1)^{\frac{s-m+1}{2}}\sum_{k=1}^{n-1}\rho_k^{n,1}\omega_k^{s+m-2}\gamma_k^{+}\delta_{ij}$
$s + m \neq 0 - even$	$(-1)^{\frac{s-m}{2}}\sum_{k=0}^{n-1}\rho_k^{n,0}\omega_k^{s+m-2}\gamma_k^{-}\epsilon_{ij}$	$(-1)^{\frac{s-m}{2}}\sum_{k=1}^{n-1}\rho_k^{n,1}\omega_k^{s+m-2}\gamma_k^{-}\epsilon_{ij}$
	$[\psi_i^{(s)}, \bar{\psi}_j^{(m)}\}$	
$s = m = 0$	$-i\sum_{k=0}^{n-1}\rho_k^{n,0}\omega_k^{-1}\gamma_k^{+}\delta_{ij}$	
$s + m - odd$	$(-1)^{\frac{s-m-1}{2}}\left(\omega_0\sum_{k=0}^{n-1}\rho_k^{n,0}\omega_k^{s+m-2}\gamma_k^{+}\delta_{ij} - i\sum_{k=0}^{n-1}\rho_k^{n,0}\omega_k^{s+m-1}\gamma_k^{-}\epsilon_{ij}\right)$	
$s + m \neq 0 - even$	$(-1)^{\frac{s-m-2}{2}}\left(i\sum_{k=0}^{n-1}\rho_k^{n,0}\omega_k^{s+m-1}\gamma_k^{+}\delta_{ij} + \omega_0\sum_{k=0}^{n-1}\rho_k^{n,0}\omega_k^{s+m-2}\gamma_k^{-}\epsilon_{ij}\right)$	

(27)

has been chosen. Above we denote $\gamma_k^{\pm} = \frac{1}{2}\left(\frac{1}{\gamma_{k,1}} \pm \frac{1}{\gamma_{k,2}}\right)$. This structure is degenerate provided $g_{n,0} = 0$ and/or $g_{n,1} = 0$, where

$$g_{n,s} = \sum_{k=s}^{n-1} \frac{\rho_k^{n,s}}{2\omega_k^2}\left(\frac{1}{\gamma_{k,1}} - \frac{1}{\gamma_{k,2}}\right) = \sum_{k=s}^{n-1} \frac{\rho_k^{n,s}\gamma_k^-}{\omega_k^2}.$$

By this reason, we restrict our consideration only to the case when $g_{n,0} \neq 0$, $g_{n,1} \neq 0$.

The generalization of the coordinates (18) reads

$$\begin{aligned} &\mathfrak{q}_{k,i} = \sqrt{|\gamma_{|k|,i}|}q_{k,i}, \quad \mathsf{z}_{s,i} = \frac{1}{\prod\limits_{m=s}^{n-1}\omega_m\sqrt{|g_{n,s}|}} z_{s,i}, \ \mathfrak{p}_{k,i} = (-1)^{k+i+1}\mathrm{sign}(\gamma_{k,i})\sqrt{|\gamma_{|k|,i}|}p_{k,i}, \\ &\Psi_{k,i} = \sqrt{|\gamma_{|k|,i}|}\vartheta_{k,i}, \ \Theta_i = \frac{1}{\prod\limits_{m=0}^{n-1}\omega_m\sqrt{|g_{n,0}|}}\theta_i, \quad \bar{\Psi}_{k,i} = (\Psi_{k,i})^*, \ \bar{\Theta}_i = (\Theta_i)^*. \text{ (no sum)} \end{aligned} \tag{28}$$

With respect to the Poisson structure (27), these variables obey the relations

$$\begin{aligned} &[\mathfrak{q}_{k,i}, \mathfrak{p}_{m,j}\} = \delta_{km}\delta_{ij}, \qquad [\mathsf{z}_{s,i}, \mathsf{z}_{m,j}\} = -\mathrm{sign}(g_{n,s})\delta_{sm}\epsilon_{ij}, \\ &[\Theta_i, \bar{\Theta}_j\} = \omega_0\,\mathrm{sign}(g_{n,0})\epsilon_{ij}, \quad [\Psi_{k,i}, \bar{\Psi}_{m,j}\} = -i\,\mathrm{sign}(\gamma_{|k|,i})\delta_{km}\delta_{ij}. \text{ (no sum)} \end{aligned} \tag{29}$$

Then the alternative Hamiltonian (26) may be rewritten as

$$\mathcal{H} = \sum_{k=-n+1}^{n-1} \frac{\mathrm{sign}(\gamma_{|k|,i})}{2}\left(\mathfrak{p}_{k,i}^2 + \omega_k^2\mathfrak{q}_{k,i}^2 + 2\omega_k\Psi_{k,i}\bar{\Psi}_{k,i}\right). \tag{30}$$

To find supercharges corresponding to this alternative Hamiltonian, let us introduce the following auxiliary action functional

$$\begin{aligned} \mathcal{S} = \frac{1}{2}\int dt \sum_{k=-n+1}^{n-1} &\mathrm{sign}(\gamma_{|k|,i})(\dot{\mathfrak{q}}_{k,i}^2 - \omega_k^2\mathfrak{q}_{k,i}^2 + i\Psi_{k,i}\dot{\bar{\Psi}}_{k,i} + i\bar{\Psi}_{k,i}\dot{\Psi}_{k,i} - 2\omega_k\Psi_{k,i}\bar{\Psi}_{k,i}) + \\ &+ \sum_{s=0}^{1}\mathrm{sign}(g_{n,s})\epsilon_{ij}\mathsf{z}_{s,i}\dot{\mathsf{z}}_{s,j} + \frac{\mathrm{sign}(g_{n,0})}{\omega_0}\epsilon_{ij}\left(\Theta_i\dot{\bar{\Theta}}_j - \bar{\Theta}_i\dot{\Theta}_j\right), \end{aligned}$$

which is invariant under the transformations

$$\begin{aligned} &\delta\mathfrak{q}_{k,i} = \Psi_{k,i}\alpha + \bar{\Psi}_{k,i}\bar{\alpha}, \qquad \delta\Psi_{k,i} = (-i\dot{\mathfrak{q}}_{k,i} + \omega_k\mathfrak{q}_{k,i})\bar{\alpha}, \\ &\delta\Theta_i = \omega_0(\mathsf{z}_{0,i} + \mathrm{sign}(g_{n,0}g_{n,1})\mathsf{z}_{1,i})\bar{\alpha}, \\ &\delta\mathsf{z}_{k,i} = \Theta_i\alpha + \bar{\Theta}_i\bar{\alpha}, \qquad \delta\bar{\Psi}_{k,i} = (-i\dot{\mathfrak{q}}_{k,i} - \omega_k\mathfrak{q}_{k,i})\alpha, \\ &\delta\bar{\Theta}_i = -\omega_0(\mathsf{z}_{0,i} + \mathrm{sign}(g_{n,0}g_{n,1})\mathsf{z}_{1,i})\alpha. \end{aligned}$$

The Noether charges associated with these symmetries read

$$\begin{aligned} \mathcal{Q} = &\sum_{k=-n+1}^{n-1}\Psi_{k,i}(\mathfrak{p}_{k,i} - i\,\mathrm{sign}(\gamma_{|k|,i})\omega_k\mathfrak{q}_{k,i}) \\ &- \epsilon_{ij}\Theta_i(\mathrm{sign}(g_{n,0})\mathsf{z}_{0,j} + \mathrm{sign}(g_{n,1})\mathsf{z}_{1,j}), \ \bar{\mathcal{Q}} = (\mathcal{Q})^*. \end{aligned}$$

These integrals of motion, together with the Hamiltonian (30), obey the following relations

$$[\mathcal{H}, \mathcal{Q}\} = 0, \qquad [\mathcal{Q}, \bar{\mathcal{Q}}\} = -2i\mathcal{H} - (\mathrm{sign}(g_{n,0}) + \mathrm{sign}(g_{n,1}))\epsilon_{ij}\Theta_i\bar{\Theta}_j, \qquad [\mathcal{H}, \bar{\mathcal{Q}}\} = 0,$$
$$[\mathcal{Q}, \mathcal{Q}\} = -(\mathrm{sign}(g_{n,0}) + \mathrm{sign}(g_{n,1}))\epsilon_{ij}\Theta_i\Theta_j,$$
$$[\bar{\mathcal{Q}}, \bar{\mathcal{Q}}\} = -(\mathrm{sign}(g_{n,0}) + \mathrm{sign}(g_{n,1}))\epsilon_{ij}\bar{\Theta}_i\bar{\Theta}_j,$$

under the bracket (29). Thus, we have one more condition on the coefficients $\gamma_{k,i}$

$$\mathrm{sign}(g_{n,0}) = -\mathrm{sign}(g_{n,1}). \tag{31}$$

It is evident that infinitely many possible sets of parameters $\gamma_{k,i}$ obey this restriction.

5. Quantization

To quantize an $\mathcal{N}=2$ supersymmetric odd-order PU oscillator with the Hamiltonian (30), let us introduce hermitian bosonic operators $\hat{\mathfrak{q}}_{k,i}, \hat{\mathfrak{p}}_{k,i}$ $\hat{\mathfrak{z}}_{s,i}$ as well as fermionic operators $\hat{\Psi}_{k,i}$, $\hat{\bar{\Psi}}_{k,i} = (\hat{\Psi}_{k,i})^\dagger$, $\hat{\Theta}_i$, $\hat{\bar{\Theta}}_i = (\hat{\Theta}_i)^\dagger$. According to (29) and (31), they obey the following nonvanishing (anti)commutation relations

$$\begin{aligned} &[\hat{\mathfrak{q}}_{k,i}, \hat{\mathfrak{p}}_{m,j}] = i\hbar\delta_{km}\delta_{ij}, && [\hat{\mathfrak{z}}_{s,i}, \hat{\mathfrak{z}}_{m,j}] = -i(-1)^s\hbar\,\mathrm{sign}(g_{n,0})\delta_{sm}\epsilon_{ij}, \\ &\{\hat{\Theta}_i, \hat{\bar{\Theta}}_j\} = i\hbar\omega_0\,\mathrm{sign}(g_{n,0})\epsilon_{ij}, && \{\hat{\Psi}_{k,i}, \hat{\bar{\Psi}}_{m,j}\} = \hbar\,\mathrm{sign}(\gamma_{|k|,i})\delta_{km}\delta_{ij}, \ (\text{no sum}) \end{aligned} \tag{32}$$

where $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ stand for the anticommutator and commutator, respectively. $\hbar$ is the reduced Planck constant.

As the next step, we may introduce the creation $\bar{a}_{k,i}$, $\bar{c}_{k,i}$ and annihilation $a_{k,i}$, $c_{k,i}$ operators, which correspond to oscillator coordinates $(\mathfrak{q}_{k,i}, \mathfrak{p}_{k,i}, \Psi_{k,i}, \bar{\Psi}_{k,i})$

$$\left.\begin{aligned} a_{k,i} &= \sqrt{\frac{|\omega_k|}{2\hbar}}\hat{\mathfrak{q}}_{k,i} + i\frac{1}{\sqrt{2|\omega_k|\hbar}}\hat{\mathfrak{p}}_{k,i}, \quad c_{k,i} = \frac{1}{\sqrt{\hbar}}\hat{\Psi}_{k,i}, \\ \bar{a}_{k,i} &= \sqrt{\frac{|\omega_k|}{2\hbar}}\hat{\mathfrak{q}}_{k,i} - i\frac{1}{\sqrt{2|\omega_k|\hbar}}\hat{\mathfrak{p}}_{k,i}, \quad \bar{c}_{k,i} = \frac{1}{\sqrt{\hbar}}\hat{\bar{\Psi}}_{k,i}, \end{aligned}\right. \Rightarrow \begin{aligned} &[a_{k,i}, \bar{a}_{m,j}] = \delta_{km}\delta_{ij}, \\ &\{c_{k,i}, \bar{c}_{m,j}\} = \mathrm{sign}(\gamma_{|k|,i})\delta_{km}\delta_{ij}. \end{aligned}$$

Thus, for negative values of $\gamma_{k,i}$, we have $\{c_{k,i}, \bar{c}_{m,j}\} = -\delta_{km}\delta_{ij}$. Taking into account the analysis in Refs. [13,36], these relations bring about negative norm states. To avoid this feature, we set all coefficients $\gamma_{k,i}$ to be positive.

For the variables $\mathfrak{z}_{s,i}$, Θ_i, and $\bar{\Theta}_i$, the creation $\bar{b}_s$, $\bar{d}_s$ and annihilation b_s, d_s operators may be defined as follows [37]

$$\begin{aligned} b_s &= \frac{1}{\sqrt{2\hbar}}(\hat{\mathfrak{z}}_{s,1} - i(-1)^s\mathrm{sign}(g_{n,0})\hat{\mathfrak{z}}_{s,2}), \\ d_s &= \frac{1}{\sqrt{2\hbar\omega_0}}(\hat{\Theta}_1 + i(-1)^s\mathrm{sign}(g_{n,0})\hat{\Theta}_2), \\ \bar{b}_s &= \frac{1}{\sqrt{2\hbar}}(\hat{\mathfrak{z}}_{s,1} + i(-1)^s\mathrm{sign}(g_{n,0})\hat{\mathfrak{z}}_{s,2}), \\ \bar{d}_s &= \frac{1}{\sqrt{2\hbar\omega_0}}(\hat{\bar{\Theta}}_1 - i(-1)^s\mathrm{sign}(g_{n,0})\hat{\bar{\Theta}}_2), \end{aligned} \qquad s = 0, 1.$$

These operators obey the following nonvanishing relations

$$[b_s, \bar{b}_p] = \delta_{sp}, \qquad \{d_s, \bar{d}_p\} = (-1)^s \delta_{sp}.$$

Unfortunately, the relation $\{d_1, \bar{d}_1\} = -1$ leads to the presence of negative norm states in the corresponding Fock space [13,36].

6. Conclusion

To summarize, in this work we have introduced an $\mathcal{N} = 2$ supersymmetric generalization for the odd-order PU oscillator with distinct frequencies of oscillation. This system is invariant under the time translations. We have observed that the corresponding integral of motion can be presented as a direct sum of one-dimensional $\mathcal{N} = 2$ supersymmetric harmonic oscillators which alternate in a sign. This representation has allowed us to construct a family of Hamiltonian structures for an $\mathcal{N} = 2$ supersymmetric odd-order PU oscillator. Unfortunately, quantization of the system revealed the presence of negative norm states in the corresponding Fock space.

Turning to further possible developments, it is worth constructing various generalizations of an $\mathcal{N} = 2$ supersymmetric odd-order PU oscillator which are compatible with the alternative Hamiltonian formulation. In particular, it would be interesting to generalize deformed odd-order PU oscillator introduced in paper [11] as well as higher-derivative field theories considered in [38] to an $\mathcal{N} = 2$ supersymmetric case. A construction of $\mathcal{N} = 2$ supersymmetric many particle higher-derivative systems is also of interest. In this context it is worth studying higher-derivative generalizations of $\mathcal{N} = 2$ supersymmetric many body models constructed in papers [39–42]. The odd-order PU oscillator with weak supersymmetry [43] has been introduced in paper [44]. It is also worth investigating a Hamiltonian formulation of this system. These issues will be studied elsewhere.

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Appendix. List of identities

When verifying the fact that the equations (7) are satisfied with respect to the Hamiltonian structures introduced in both Sects. 2 and 4, the following identities

$$P_{2k}^{n,s} = \sum_{p=s}^{n-1} (-1)^{n+p+1} \omega_p^{2n+2k-2s-2} \rho_p^{n,s}, \qquad k = -n+s+1, -n+s+2, ..;$$

$$\sum_{k=s}^{n-1} (-1)^{k+s} (-\omega_k^2)^r \sigma_{p,k}^{n,s} \rho_k^{n,s} = \begin{cases} \delta_{rp}, & r = 0, 1, .., n-s-1; \\ -\sigma_p^{n,s}, r = n-s; \end{cases}$$

$$\sum_{r=0}^{n-1} \frac{(-1)^r}{\omega_r^2} \sigma_{p,r}^{n,0} \rho_r^{n,0} = \frac{\sigma_{p+1}^{n,0}}{\prod_{k=0}^{n-1} \omega_k^2};$$

$$\sum_{r=0}^{n-s-1}(-1)^r\omega_q^{2r}\sigma_{r,k}^{n,s}=\frac{(-1)^{k+s}}{\rho_k^{n,s}}\delta_{qk};$$

$$\sigma_{p,k}^{n,s}=\sum_{r=0}^{n-p-s-1}(-1)^r\omega_k^{2r}\sigma_{p+r+1}^{n,s},\qquad k=s,s+1,..,n-1;$$

with

$$\sigma_{m,k}^{n,s}=\sum_{\substack{i_1<i_2<..<i_{n-m-1}=s\\ i_1,i_2,..,i_{n-m-1}\neq k}}^{n-1}\omega_{i_1}^2\omega_{i_2}^2..\omega_{i_{n-m-1}}^2,\qquad \sigma_{n-s-1,k}^{n,s}\equiv 1,$$

prove to be helpful. The proofs of these identities can be found in Refs. [10,11,45].

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