

Calculation by iterative method of linear viscoelastic plate under biaxial tension

A A Svetashkov¹, J Miciński², K K Manabaev¹ and A A Vakurov¹

¹ Tomsk Polytechnic University, 30, Lenina Ave., Tomsk, 634050, Russia

² University of Warmia and Mazury in Olsztyn Faculty of Animal Bioengineering Department of Cattle Breeding and Milk Evaluation, 5, Oczapowskiego Ave., Olsztyn, 10-957, Poland

E-mail: mkk@tpu.ru

Abstract. In this paper, we used the iterative solution algorithm, proposed in the work of Pavlov and Svetashkova. This algorithm results in a complete separation of spatial and temporal variables, if we set up the boundary loads and (or) volumetric forces in the same kind. In this paper, we have examined the stress-strain state of a viscoelastic plate, and the results of the calculation displacements, stresses are given. In addition, we made a comparison of the calculation indices rate of convergence for the iterative process with their theoretical values.

1. Introduction

The problem of algorithmization for solving boundary tasks of linear viscoelasticity in case the constitutive equations are given in an integral form is actual. Storage of the full temporal history of changing the stress-strain state previous to time t is essential in solving this problem when using traditional approaches [1-3]. Thus, the obtainment of information about stresses and strains at time t requires the information on their changes in prior times $\tau < t$.

The application of the iterative algorithm in the calculation of tasks linear viscoelasticity allows circumventing the given problem, because implementation of this method involves the separation of spatial and temporal variables. This, in turn, eliminates the necessity of storage of information about the history of loading the construction.

Implementation of the new computational method of the iterative algorithm in the environment of the software complex of the finite element method is the aim of this work. The method is implemented by the example of solving a boundary problem about biaxial loading of the viscoelastic plate. We compare the indicators of rate convergence which were obtained by the theoretical and calculation methods.

2. The problem about loading plate

We consider the solution of the problem of a biaxial stretching linear viscoelastic plate under the action of the uniformly distributed loads applied to the ends of T_3 , T_4 and directed along axes x and y . The plate has been fixed at sides T_1 , T_2 , figure 1.

The equilibrium equations during displacements of flat problem linear viscoelasticity have the form:



$$\left(K_0 + \frac{1}{3} G^* \right) \theta_{,\alpha} + G^* \Delta u_\alpha = 0, \alpha = 1, 2 \tag{1}$$

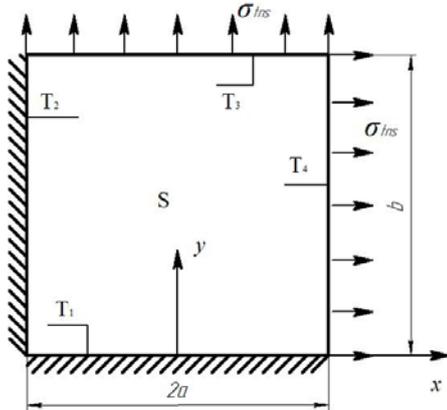


Figure 1. Scheme of loaded plate.

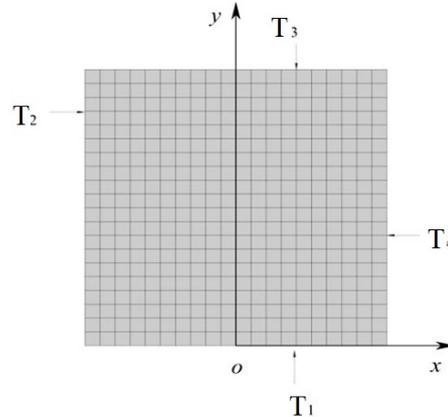


Figure 2. Image of finite – element model plate.

The boundary conditions in stresses are:

$$\begin{cases} \sigma_{11}l + \tau_{12}m = X_n \\ \sigma_{22}m + \tau_{21}l = Y_n, \end{cases} \tag{2}$$

here, $\theta = u_{1,1} + u_{2,2}$ – volumetric deformation; comma before the index corresponds to differentiation with respect to the spatial coordinate (for example, $u_{1,1} = \frac{\partial u_1}{\partial x}$); $u_\alpha, \alpha = 1, 2$ – components of displacement vector \bar{u} ; Δ – Laplace operator; $\sigma_{\alpha\beta}$ – components of the stress tensor; l, m – direction cosines of the normal to the boundary contour; X_n, Y_n – boundary loads; G^* – integral operator of relaxation.

$$G^* \varepsilon_{\alpha\beta} \equiv \int_0^t R(t-\tau) d\varepsilon_{\alpha\beta}(\tau), \tag{3}$$

where $R(t)$ – function of the shear relaxation, $R(0) = G_0$; G_0, K_0 – elastic-instantaneous modules of shear and volumetric compression; $\varepsilon_{\alpha\beta}$ – components of the deformation tensor. The physical equations of linear viscoelastic body have the form:

$$\sigma_{\alpha\beta} = \left(K_0 - \frac{2}{3} G^* \right) \theta \delta_{\alpha\beta} + 2G^* \varepsilon_{\alpha\beta}, \delta_{\alpha\beta} = \begin{cases} 1, \alpha = \beta, \\ 0, \alpha \neq \beta. \end{cases} \tag{4}$$

We will solve system integral-differential equations (1), (2) by the iterative method, presented in the work of Pavlov and Svetashkov [4]. For this purpose, we introduce the auxiliary constitutive equations:

$$\sigma_{\alpha\beta} = G^* \left[\left(k_s - \frac{2}{3} \right) \theta \delta_{\alpha\beta} + 2\varepsilon_{\alpha\beta} \right] \tag{5}$$

where k_s is a constant, $k_s \in [1, K_0 / G_0]$.

Constitutive equations (5) correspond to a viscoelastic material with proportional operators of volumetric and shearing relaxation. In this case, as Christensen [5] shown, the stress-strain state of the structure varies in proportion to a function of time.

We will solve boundary problems (1), (2) by replacing constitutive equations (4) with equations (5), and we will represent the discrepancy as fictitious volumetric and surface forces. As a result, in each iteration, we will obtain the iterative type equations:

$$G^* \left[\left(k_s + \frac{1}{3} \right) \theta_{,\alpha}^{n+1} + \Delta u_{\alpha}^{n+1} \right] = (k_s G^* - K^*) \theta_{,\alpha}^n, \tag{6}$$

$$\sigma_{\alpha\beta}^0 \left(\bar{u}^{-n+1} \right) n_{\beta} = S_{\alpha}^0 + \left[\sigma_{\alpha\beta}^0 \left(\bar{u}^{-n} \right) - \sigma_{\alpha\beta} \left(\bar{u}^{-n} \right) \right] n_{\beta}, (\alpha, \beta = 1, 2), n = 0, 1, \dots$$

Here $S_1^0 = X_n, S_2^0 = Y_n$. Then, for the n -th approximation, we will have:

$$u_{\alpha}^{(n)}(x, y, t) = \sum_{i=1}^n \tilde{u}_{\alpha}^{(i)}(x, y) \varphi_{i-1}(t),$$

$$\varphi_i(t) = \left[\bar{G}^{*-1} (k_s G^* - K_0) \right]^i \varphi_0(t), \tag{7}$$

$$\varphi_0(t) = \bar{G}^{*-1} H, \bar{G}^* = G^* / G_0.$$

Stresses in viscoelastic body are defined as:

$$\sigma_{\alpha\beta}^{(n)} = \sigma_{\alpha\beta}^{(n-1)} + \tilde{\theta}^{(n)} \left(K_0 - \frac{2}{3} G^* \right) \varphi_{n-1} \delta_{\alpha\beta} + 2 \tilde{\mathcal{E}}_{\alpha\beta}^{(n)} G^* \varphi_{n-1}, n = 1, 2, \dots \tag{8}$$

Functions $\tilde{u}_{\alpha}^{(i)}(x, y)$ are defined as the solution of elastic problems.

3. Analysis of the results

The numerical calculation of the elastic problem has been implemented in software package ANSYS, the calculated area of the finite – element model is presented in Figure 2. The constants of material are: $G_0 = 2$ MPa, $K_0 = 10$ MPa . The boundary loads equal $X_n = Y_n = \sigma_{ms} = 15$ MPa .

According to (7), the results of the viscoelastic solution of the problem for the average speed of relaxation and specified parameters $\lambda = 0.04, \gamma = 0.01, k_s = 4.5$, as well as when parameter η that determines the ratio of the resiliently instant module to prolonged module $\eta = G_0 / G_{\infty}$ takes a value equal to 5, $t = 100$ min, are shown in Table 1.

Table 1. Results $u_{\alpha}^{(i)}(x, y, t)$ of plate calculation

n	$\tilde{u}_1^{(i)}$, cm	$\tilde{u}_2^{(i)}$, cm	$\varphi_{n-1}(t)$	$u_1^{(i)}$, cm	$u_2^{(i)}$, cm
1	1.621E-02	1.692E-02	4.107	0.066	0.069
2	3.613E-03	3.723E-03	-3.147	-0.011	-0.011
3	8.504E-04	8.618E-04	2.372	0.002	0.002
4	2.108E-04	2.095E-04	-1.748	-3.669E-04	-3.669E-04
5	5.406E-05	5.314E-05	1.245	6.726E-05	6.612E-05
6	1.437E-05	1.392E-05	-0.840	-1.204E-05	-1.172E-05
7	3.880E-06	3.754E-06	0.514	2.000E-06	1.932E-06
8	1.070E-06	1.038E-06	-0.252	-2.694E-07	-2.590E-07

For assessment of coefficient k_s impact on the rate of the convergence process, we consider the following cases of its assignment:

$$1) k_s = 0.7\tilde{k}_s; 2) k_s = 0.85\tilde{k}_s; 3) k_s = 0.9\tilde{k}_s, \tag{9}$$

where $\tilde{k}_s = K_0 / G_0 = 5$.

In the future, we will consider the calculated indicators of convergence speed, which are determined by the formula:

$$q_n = \frac{W_n(u^{n+1} - u^n)}{W_n(u^n - u^{n-1})}, \quad u^n = (u^n, v^n), n = 1, 2, \dots$$

Here $W_n(u^n)$ – specific potential energies of stresses and deformations.

$$W_n(u^n) = \frac{1}{2}(\sigma_{\alpha\beta}^n \varepsilon_{\alpha\beta}^n)$$

Functions $W_n(u^n)$ have been calculated at each time point for the selected element of the grid calculated model of the plate. Next, we determine the total calculated indicator of degree compression for n iterations

$$q = \prod_{k=1}^n q_k \tag{10}$$

This indicator expresses the ratio of convolution, calculated for the difference of n and $(n-1)$ approximations, to the respective convolution, calculated for difference of the first and zero approximation.

We have given Figure 3 for the assessment of impact k_s on the rate of convergence solution. The logarithmic dependences for the total indicator of degree compression q , defined by (10), depending on the time and iterations, are also presented in Figure 3

The logarithmic dependences of the theoretical \bar{q} and calculated q (10) indicators of the convergence rate of the iterative solution depending on number n of steps for procedure solution are shown in Figure 4.

The results of solving the problem are given for the average rate of relaxation and the set parameters: $\lambda = 0.04$; $\gamma = 0.01$, $\eta = 5$, $k_s = 0.9\tilde{k}_s$. The dashed curves are obtained by the iterative method, the continuous curves – theoretically. The time interval of 50 minutes corresponds to curves 1, 5; 100 min – curves 2, 6; 1000 min – curves 3, 7; 1500 min – curves 4, 8.

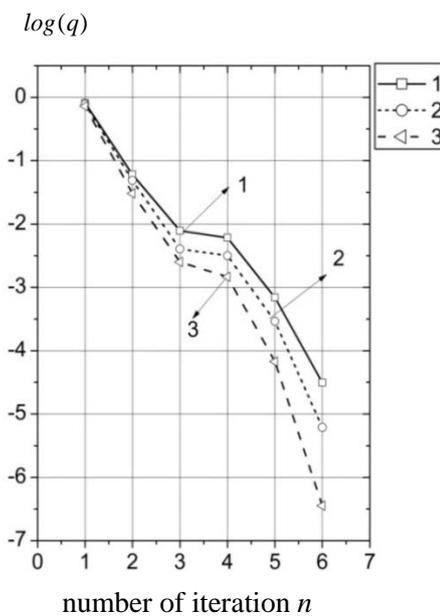


Figure 3. The logarithmic dependences for q from the number of iteration, $t = 1000$ min

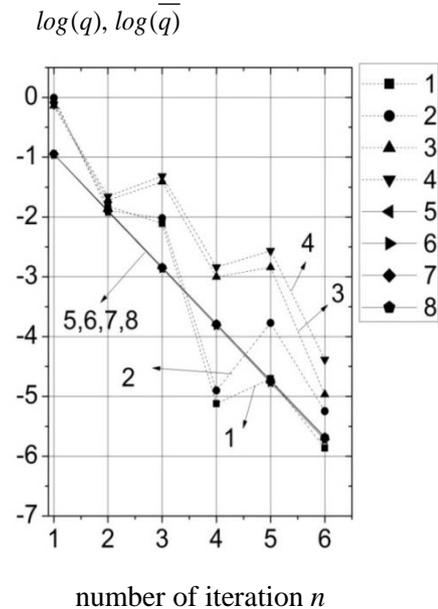


Figure 4. The dependence of indicators for the rate of convergence and n .

The curve of stresses σ_{11} , σ_{22} (biaxial tension by equal efforts) depending on the time is shown in Figure 5.

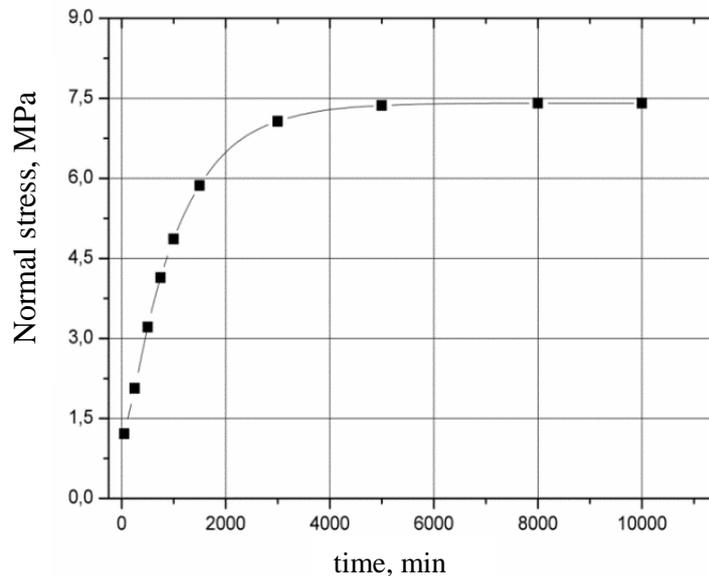


Figure 5. The dependence of stresses σ_{11} , σ_{22} on time.

The curve of stresses values σ_{22} depending on the number of iteration is presented in Figure 6. The time was chosen to be 200 minutes.

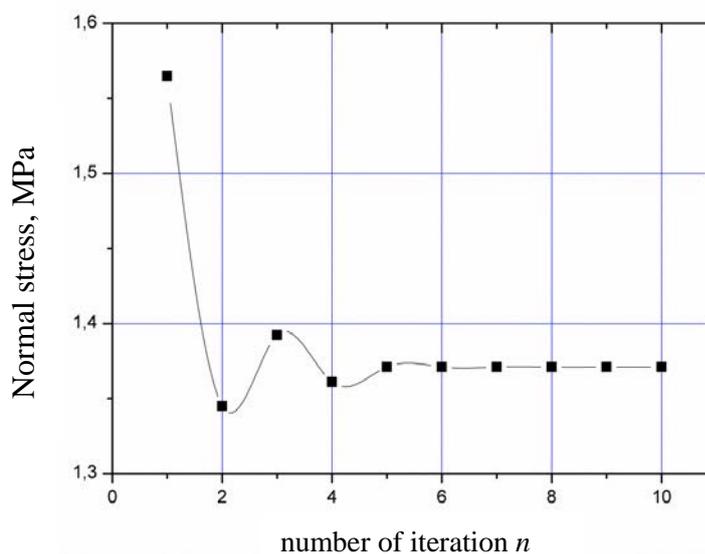


Figure 6. The curve of convergence stress σ_{22} depending on the number of iterations.

4. Conclusion

The numerical solution of the plane problem of linear viscoelasticity theory [8 – 10] as applied to the calculation rectangular plate (biaxial tension) is presented in the work. The following results were obtained:

a) the iterative algorithm which is used in the work, gives a fairly rapid convergence (for stresses, the full convergence is achieved after ten iterations, for displacements after eight);

- b) the outlays of machine time are the same for any preassigned moment of the loading history plate. This fact is an advantage for the used iterative method;
- c) comparison of the calculated and theoretical indicators of convergence for the iterative algorithm showed their convergence within the limits of 10%.
- d) we investigated the effect of arbitrary numerical parameter k_s included in the auxiliary physical equations of the communications law of viscoelastic stresses and deformations on the convergence of the iterative process.

References

- [1] Chazal C, Pitti R M 2001 *J. of Theor. and Appl. Mech.* **49** 1029-1048
- [2] Matveenko V, Trufanov N 2013 *J. of Eng. Math.* **78** 119-129
- [3] Ataoglu S 2009 *Struct. Eng. and Mech.* **32** 407-427
- [4] Pavlov S M and Svetashkov A A 1993 *Russ. Phys. J.* **36** 400-406
- [5] Christensen R M 1979 *Mechanics of composite materials* (New York: Academic Press)
- [6] Svetashkov A A, Kupriyanov N A and Manabaev K K 2013 *Russian Physics J.* 209-211
- [7] Kupriyanov N A, Simankin F A and Manabaev K K 2016 *IOP Conf. Ser.: Mat. Sci. and Eng.* (Tomsk) vol 124 (IOP Publishing) 012100
- [8] Svetashkov A A, Vakurov A A 2016 *IOP Conf. Ser.: Mat. Sci. and Eng.* (Tomsk) vol 124 (IOP Publishing) 012099
- [9] Svetashkov A A 2000 *Mechanics of Composite Materials* **36** 37-44
- [10] Svetashkov A A, Kupriyanov N A and Manabaev K K 2016 *Key Eng. Mat.: Sci. J.* **685** 206-210