Non-Markovian dynamics of fully coupled fermionic and bosonic oscillators

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The non-Markovian Langevin approach is applied to study the dynamics of fermionic (bosonic) oscillator linearly coupled to a fermionic (bosonic) environment. The analytical expressions for occupation numbers in two different types of couplings (rotating-wave approximation and fully coupled) are compared and discussed. The weak-coupling and high- and low-temperature limits are considered as well. The conditions under which the environment imposes its thermal equilibrium on the collective subsystem are discussed. The sameness of the results, obtained with both the Langevin approach and the discretized environment method are shown. Short- and long-time nonequilibrium dynamics of fermionic and bosonic open quantum systems are analyzed both analytically and numerically.

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I. INTRODUCTION

A goal of nonequilibrium statistical mechanics is to make good, yet simple, models of complicated phenomena out of equilibrium and to analyze them. Investigations of dissipative quantum non-Markovian subsystems beyond the weak-coupling or high-temperature limits request exactly solvable models [1–24] to be widely applicable. This class of models has been considered to investigate various aspects in mesoscopic physics where a given two-level system interacts with a bosonic or fermionic environment modeled by a set of harmonic oscillators. It is convenient to describe an influence of the environment on the system using a spectral density which contains information about the spectrum of the environment as well as the frequency-dependent coupling [11,12,19,25]. The complete information about the effect of the thermal bath is encapsulated in the single spectral function [1,2,7]. The systems with fermionic baths are of interest due to the possibility of creating and manipulating rather small fermionic systems in various fields of physics [7,11,12,17–19,26]. Models describing the interaction between fermionic systems and spin degrees of freedom play an important role [7,11,12,17–19,26]. In particular, an approach of these systems to equilibrium would greatly help to understand how they can reach the thermodynamic limits and how the thermalizations of the isolated system and the collective subsystem of this isolated system are related [21,27]. It is interesting to study the crossover from coherent to incoherent dynamics in the damped quantum system [29].

Recently, several stochastic methods have been proposed to consider the problem of a system coupled to an environment. This includes the functional-integral approach [7,11,12,17–19,26], the quantum state diffusion approach [30–33], quantum jumps [34], the quantum Langevin approach [20–24,35], the quantum Monte Carlo approach [36–39], or the stochastic method of Refs. [19,40]. Applications of stochastic methods to non-Markovian open quantum systems are progressing but still remain tedious when the complexity of the system increases. So, further development of theoretical methods is required.

In Ref. [41], we considered the quadratic fermionic Hamiltonians for collective and internal subsystems linearly coupled within the rotating-wave approximation (RWA) to analyze the role of the fermionic statistics (in the comparison with the bosonic statistics) in the dynamics of the collective motion. The Langevin approach [20–24] was applied to find the effects of fluctuations and dissipations in macroscopic systems. The Langevin method in the kinetic theory significantly simplifies the calculation of nonequilibrium quantum and thermal fluctuations and provides a clear picture of the dynamics. Many problems in various open quantum systems can be described by using the Langevin equations in the space of relevant collective coordinates. The results of the Langevin approach were confirmed by considering the dynamics of an open quantum system with the discretized environment method (DEM) [28] which allows us to incorporate environment explicitly in a discretized form.

The aim of the present work is to extend the results of Ref. [41] and consider the general case of a fully coupled (FC) oscillator modeling fermionic (bosonic) collective subsystem coupled with a fermionic (bosonic) heat bath. The results obtained will be checked with the DEM [28] extended to treat FC oscillator and compared with those obtained in the RWA case.

In Secs. II and III, the model is formulated and the expressions for occupation numbers are obtained. The asymptotic occupation numbers are discussed in Sec. IV. The weak-coupling limit is considered in Sec. V and the results are summarized in Sec. VI.

II. MODEL HAMILTONIAN OF FULLY COUPLED FERMIONIC OSCILLATOR

We consider a two-level fermionic system (collective subsystem) with creation $a^\dagger$ and annihilation $a$ operators, and with frequency $\hbar\omega$. This system interacts with a bath consisting of two-level fermionic systems, labeled by index $\nu$, with creation and annihilation operators $a_\nu^\dagger$ and $a_\nu$, and frequency $\hbar\omega_\nu$, respectively. For two-level fermionic systems, the operators satisfy the following permutation relations:

$$aa^\dagger + a^\dagger a = 1, \quad aa^\dagger a^\dagger a = a^\dagger a^\dagger a^\dagger a = 0,$$

$$a_\nu a_\nu^\dagger + a_\nu^\dagger a_\nu = a_\nu^\dagger a_\nu^\dagger a_\nu = 0,$$

$$a_\nu a_\nu^\dagger a_\nu = a_\nu a_\nu^\dagger a_\nu = 0.$$
The Hamiltonian of the whole system is
\[ H = H_c + H_b + H_{cb}, \]
where
\[ H_c = \hbar \omega a a^+ \]
is the Hamiltonian of isolated collective subsystem, and
\[ H_b = \sum_v \hbar \omega_v a_v a_v^+ \]
is the Hamiltonian of the bath. As the detailed analysis of the dynamics of occupation numbers in the case of RWA coupling between the system and bath was presented in Ref. [41], here we study the FC-type coupling. In this case, the interaction Hamiltonian is written as
\[ H_{cb} = \sum_v g_v (a_v^+ a + a_v a^+). \]
The real constants \( g_v \) in Eq. (5) determine the coupling strength between the collective and bath “\( v \)” subsystems.

III. EXPRESSIONS FOR OCCUPATION NUMBER OF COLLECTIVE SUBSYSTEM

By commuting the creation and annihilation operators of the collective subsystem with total Hamiltonian \( H \), one can obtain the Heisenberg equations of motion for corresponding operators:
\[
\frac{d}{dt} a_v^+ = i \omega a_v^+ + (1 - 2a_v^+ a) \frac{i}{\hbar} \sum_v g_v (a_v^+ + a_v),
\]
\[
\frac{d}{dt} a_v = -i \omega a - (1 - 2a^+ a) \frac{i}{\hbar} \sum_v g_v (a_v^+ + a_v). \quad (6)
\]
These equations contain terms proportional to \( 2a^+ a \) and could not be solved analytically. However, keeping the terms with \( 2a^+ a \) in Eqs. (6), we obtain the zero operators \( a^+ a \) and \( a^+ a \) in the equation of motion for the occupation number \( n_v(t) = a^+ (t) a(t) \). As follows from Eqs. (1), one should skip the terms \( a^+ a \) and \( a a^+ \) in the equation of motion for \( n_v(t) \) (see Appendix A). Because our aim is to derive and study \( n_v(t) \), we disregard the terms proportional to \( 2a^+ a \) in Eqs. (6) (Appendix A). Note that, for bosonic systems, the equations for the creation and annihilation operators coincide with Eqs. (6) without the terms proportional to \( 2a^+ a \). The procedure for obtaining the occupation number of collective subsystem is well established [23,41]. The details related to the FC oscillator are given in Appendix B. Here, we directly write the final expression for the time dependence of occupation number for fermionic (f) and bosonic (b) collective subsystem:
\[
\begin{align*}
n_{1b}(t) & = A^*(t) A(t) n_{1b}^c(0) + B^*(t) B(t) \left[ 1 \mp n_{1b}^c(0) \right] + \Gamma_{1b}(t) + J_{1b}(t), \\
\end{align*}
\]
and
\[
J_{1b}(t) = \frac{g_0}{\pi} \int_0^\infty dw \frac{\gamma^2 w}{\gamma^2 + w^2} \left[ 1 \mp n_{1b}(w) \right] N^*(w,t) N(w,t). \quad (9)
\]
The upper sign in Eqs. (7) and (9) corresponds to the fermionic subsystem with the \( n_v(w) = \{ \text{exp}[\hbar w/(kT)] + 1 \}^{-1} \) equilibrium Fermi–Dirac distribution, and the lower sign is related to bosonic subsystem (FC bosonic oscillator) with the \( n_v(w) = \{ \text{exp}[\hbar w/(kT)] - 1 \}^{-1} \) equilibrium Bose–Einstein distribution. Here, an Ohmic spectral density with a Drude–Lorentz cutoff is used (see Appendix B). The similarity of expressions for the occupation numbers for fermionic and bosonic systems is the consequence of similarity of the equations of motion for creation and annihilation operators. The integrals in Eqs. (8) and (9) arise when we replace the sums over \( v \) by integrals and introduce the bandwidth \( \gamma \) of the bath. The memory time \( \gamma^{-1} \) of the dissipation is the inverse of the bandwidth of the bath excitations which are coupled to the collective subsystem.

For the \( A(t), B(t), M(w,t), \) and \( N(w,t) \) the following expressions are obtained (see Appendix B):
\[
A(t) = i \frac{1}{2} \sum_{k=1}^3 \beta_k e^{i\eta k t} \times \left[ (s_k - s_0) [ (s_k + \gamma) (\Omega - \omega - 2i s_k) + 2g_0 s_k \gamma ] \right],
\]
\[
B(t) = i \frac{1}{2} \sum_{k=1}^3 \beta_k e^{i\eta k t} \times \left[ (s_k - s_0) [ (s_k + \gamma) (\Omega - \omega) + 2g_0 s_k \gamma ] \right],
\]
\[
M(w,t) = \left\{ \sum_{k=0}^3 \beta_k e^{i\eta k (s_k + \gamma) (s_k + i\omega)} \right\}^2,
\]
\[
N(w,t) = \left\{ \sum_{k=0}^3 \beta_k e^{i\eta k (s_k + \gamma) (s_k - i\omega)} \right\}^2, \quad (10)
\]
where \( s_0 = i w \) and \( s_1, s_2, \) and \( s_3 \) are the roots of the following equation:
\[
(s + \gamma) (s^2 + \omega \Omega) + 2g_0 \gamma \alpha s = 0, \quad (11)
\]
and
\[
\beta_k = \prod_{i=0}^3 \frac{1}{(s_k - s_i)}, \quad k \neq i. \quad (12)
\]
In Eq. (11), \( \Omega = \omega - 2g_0 \gamma \) is the renormalized collective frequency which is the result of coupling between the collective and internal subsystems (see Appendix B). As one of the main goals of our study is to elucidate the difference between FC and RWA oscillators, here we present also the expression for time dependence of the occupation number in the case of RWA coupling [41]:
\[
\Gamma_{1b}^{\text{RWA}}(t) = C^*(t) C(t) n_{1b}^{\text{RWA}}(0) + \Gamma_{1b}^{\text{RWA}}(t), \quad (13)
\]
with
\[
\Gamma_{1b}^{\text{RWA}}(t) = \frac{g_0}{\pi} \int_0^\infty dw \frac{\gamma^2 w}{\gamma^2 + w^2} n_{1b}(w) L^*(w,t) L(w,t). \quad (14)
\]
where
\[ C(t) = e^{i\omega t} \frac{z_1 + \gamma}{z_1 - z_2} + e^{i\omega t} \frac{z_2 + \gamma}{z_2 - z_1}, \]
\[ L(w, t) = e^{i\omega t} \frac{z_1 + \gamma}{(z_1 - z_2)(z_1 - i\omega)} \]
\[ + e^{i\omega t} \frac{z_2 + \gamma}{(z_2 - z_1)(z_2 - i\omega)} \]
\[ + e^{i\omega t} \frac{i\omega + \gamma}{(i\omega - z_1)(i\omega - z_2)}, \]
(15)
and \( z_1 \) and \( z_2 \) are the roots of the equation
\[ (z + \gamma)(z - i\Omega) - ig_0\gamma z = 0. \]
(16)

Note that, in Eq. (16), the renormalized frequency is \( \Omega = \omega - g_0\gamma \).

First, we note the structural difference of \( n_{\text{FC}}^c(t) \) and \( n_{\text{RWA}}^c(t) \). In the FC case there are two additional terms \( B^*(t)B(t)[1 \mp n_{\text{FC}}^c(0)] \) and \( J_{\text{FC}}^c(t) \) which have no analogy in the RWA case. While in the RWA case the dynamics of occupation numbers of initially unoccupied states \( [n_{\text{RWA}}^c(0) = 0] \) is determined by the integral term only, in the FC case the behavior of occupation number is determined by both integral and nonintegral terms [due to the presence of the \( B^*(t)B(t)[1 \mp n_{\text{FC}}^c(0)] \) term]. As found, the contribution of the \( B^*(t)B(t)[1 \mp n_{\text{FC}}^c(0)] \) term to \( n_{\text{FC}}^c(t) \) is especially important in the short-time range.

The second difference between the FC and RWA oscillators is related to the difference between Eqs. (11) and (16) which determine the number of the roots. In FC coupling, the occupation numbers depend on three roots, while in the RWA case there are only two roots.

In the considered range of parameters \( \gamma, \Omega, \) and \( g_0 \), Eq. (11) has one negative real root \( s_1 \) and two complex-conjugated roots \( s_2 = s_2^* \) with negative real parts and \( \text{Re}[s_1] < \text{Re}[s_2] \). This type of structure of the roots leads to exponential damping of time evolution of the functions \( A^*(t)A(t) \) and \( B^*(t)B(t) \) proportional to \( e^{i(s_2 + s_2^*)t} \).

For the RWA oscillator, Eq. (16) has two complex roots \( z_1 \) and \( z_2 \) with negative real parts. In this case \( C^*(t)C(t) \sim e^{i(s_1 + s_1^*)t} \) where \( z_2 \) is the root with the maximal real part: \( \text{Re}[z_1] < \text{Re}[z_2] \). In Fig. 1, we compare the dependence of \( s_2 + s_2^* \) and \( z_2 + z_2^* \) from Eq. (11) with \( z_2 + z_2^* \) from Eq. (16) on parameter \( \gamma \) for different coupling strengths \( g_0 \). In the RWA case, this dependence is rather weak for all \( g_0 \), while in the FC case the value of \( s_2 + s_2^* \) increases with \( \gamma \). The dependence of \( s_2 + s_2^* \) on \( \gamma \) becomes stronger with increasing \( g_0 \).

The behavior of the roots is reflected in time dependencies of \( A^*(t)A(t), B^*(t)B(t), \) and \( C^*(t)C(t) \) (Fig. 2). In the FC case, the functions \( A^*(t)A(t) \) and \( B^*(t)B(t) \) fall off stronger with time than the function \( C^*(t)C(t) \) in the RWA case. This means that the initially occupied state \( [n_{\text{RWA}}^c(0) \neq 0] \) more strongly influences the dynamics of occupation numbers in the case of the RWA oscillator.

In Fig. 3, we show the time dependencies of integral terms \( I^c_{\text{FC}}(t), J^c_{\text{FC}}(t), \) and \( I^c_{\text{RWA}}(t) \) in the expressions for occupation numbers of fermionic and bosonic collective subsystems. After some transient time the occupation numbers reach their equilibrium values. As expected, in the case of the FC oscillator we get a stronger damping of oscillations and correspondingly reach the asymptotic values more rapidly for both fermions and bosons. It is interesting to mention that \( J^c_{\text{FC}}(t) \) and \( J^c_{\text{RWA}}(t) \) are almost the same. Moreover, at the values of \( g_0 \) and \( T \) considered, these integral terms mainly contribute to \( n_{\text{FC}}^c(t) \). The role of \( J^c_{\text{FC}}(t) \) and \( J^c_{\text{RWA}}(t) \) becomes smaller with increasing

FIG. 1. The dependence of \( s_2 + s_2^* \) and \( z_2 + z_2^* \) on the parameter \( \gamma \) for the indicated coupling constant \( g_0 \). The results of calculations for FC and RWA couplings are presented by solid and dashed lines, respectively.

FIG. 2. The time dependence of indicated functions at \( g_0 = 0.1, \gamma/\Omega = 12 \).
FIG. 3. The time dependencies of integral terms in $n_{FC}f(b)(t)$ [Eq. (7)] and
$n_{RWA}f(b)(t)$ [Eq. (13)]. The calculations are performed at $g_0 = 0.1$, $\gamma/\Omega = 12,$
and $kT/\hbar/\Omega = 0.1$.

temperature or with decreasing $g_0$. The transient time is almost
independent of the statistical nature of the bath.

In Fig. 4, we show the time dependencies of occupation
number and the value of $-(dn_{FC}f(b)/dt)/n_{FC}f(b)$ of the initially
occupied state $n_{FC}f(b)(0) = 1$. The occupation numbers $n_{FC}f(b)(t)$
reach their equilibrium values faster than $n_{RWA}f(b)(t)$ for both
fermionic and bosonic collective subsystems. However, the
oscillation before reaching the asymptotes are stronger in the
case of a FC oscillator. So, the time behavior of occupation
numbers is mostly determined by the kind of coupling rather
than the nature of the system.

IV. DIFFERENTIAL EQUATION
FOR OCCUPATION NUMBER

By using the explicit time dependence of $n_{FC}f(b)(t)$ [Eq. (7)],
we derive the following differential equation:

$$
\frac{d}{dt}n_{FC}f(b)(t) = -2\lambda_{FC}^{\text{eq}}(t)n_{FC}^{\text{eq}}(t) + 2D_{FC}^{\text{eq}}(t),
$$

where

$$
\lambda_{FC}^{\text{eq}}(t) = -\frac{1}{2} \frac{d}{dt} \ln \left[ A^*(t)A(t) \mp B^*(t)B(t) \right],
$$

and

$$
D_{FC}^{\text{eq}}(t) = \lambda_{FC}^{\text{eq}}(t) \left[ B^*(t)B(t) + I_{FC}^{\text{eq}}(t) + J_{FC}^{\text{eq}}(t) \right]
+ \frac{1}{2} \frac{d}{dt} \left[ B^*(t)B(t) + I_{FC}^{\text{eq}}(t) + J_{FC}^{\text{eq}}(t) \right].
$$

In Eq. (18) the upper (lower) sign corresponds to fermionic (bosonic) subsystems.

The same type differential equation was obtain for the RWA
oscillator [41], where

$$
\lambda_{RWA}^{\text{eq}}(t) = -\frac{1}{2} \frac{d}{dt} \ln \left[ C^*(t)C(t) \right],
$$

and

$$
D_{RWA}^{\text{eq}}(t) = \lambda_{RWA}^{\text{eq}}(t)I_{RWA}^{\text{eq}}(t) + \frac{d}{dt}I_{RWA}^{\text{eq}}(t)
$$

are the time-dependent coefficients.

Comparison with discretized environment method

As a cross-check of the analytical solutions given above,
we also applied the discretized environment method [28]
generalized to treat the FC case. The simple idea developed
in this method is that the continuous environment can
be accurately discretized so that the equations of motion are
solved by a direct diagonalization of the complete (collective
subsystem + environment) Hamiltonian. Starting from the
equation of motion (6) and using the same approximation as in previous section, the evolution can be rewritten as

\[ i\hbar \frac{d}{dt} \begin{pmatrix} C \\ C^\dagger \end{pmatrix} = L \begin{pmatrix} C \\ C^\dagger \end{pmatrix}, \tag{22} \]

where \( C' = (a_{\alpha}, \{a_i\}_{\alpha=1,\ldots,N}) \) contains the annihilation operators of the collective subsystem and bath. \( N \) is the number of states used to discretize the environment. The discretization procedure is described in Ref. [28] and we only give below novel aspects associated with the FC case compared with the RWA case. \( L \) is a \( 4 \times 4 \) block matrix with

\[ L = \begin{pmatrix} L_{00}^- & L_{00}^+ \\ L_{-0}^- & L_{-0}^+ \end{pmatrix}, \tag{23} \]

where each matrix is of size \((N+1) \times (N+1)\). Using the convention that \( \nu = 0 \) labels the operator associated with the collective subsystem, we have

\[
\begin{align*}
L_{00}^- &= h\omega, & L_{v\nu}^- &= h\omega_v, & L_{00}^- &= L_{v\nu}^- = g_v, \\
L_{00}^+ &= -h\omega, & L_{v\nu}^+ &= -h\omega_v, \\
L_{00}^+ &= L_{v\nu}^+ = -g_v, & L_{00}^+ &= +g_v,
\end{align*}
\tag{24}
\]

while other matrix elements are zero. The matrix \( L \) corresponds to the generalized Hamiltonian found for superfluid systems. It could be diagonalized by using a standard Bogoliubov transformation [42]. The RWA case is recovered simply if \( L^{--} = 0 \). The diagonalization gives new operators given by

\[
\begin{align*}
A_\alpha &= \sum_\lambda U_{\alpha\lambda}^g C_\lambda - V_{\alpha\lambda}^g C_\lambda^\dagger, \\
A_\alpha^\dagger &= \sum_\lambda U_{\alpha\lambda}^g C_\lambda^\dagger - V_{\alpha\lambda}^g C_\lambda,
\end{align*}
\tag{25, 26}
\]

where \( U \) and \( V \) are the usual Bogoliubov matrices. The new operators evolve through

\[ A_\alpha(t) = e^{i\Omega_\alpha t} A_\alpha(0), \tag{27} \]

where \( \Omega_\alpha \) are the eigenvalues of \( L \). Using this representation, as in Ref. [28], one can easily obtain the expectation values of the occupation probability for the environment. An illustration of the results of the discretized environment method is given in Fig. 5 and compared with the Langevin-approach result. The alternative direct solution provided by the discretized environment method validates the different analytical results given in this paper. Below, we only show the quantum Langevin-approach result.

V. ASYMPTOTIC OCCUPATION NUMBERS

Because the roots of Eq. (11) have negative real parts, \( A^x(t \to \infty)A(t \to \infty) = 0 \), \( B^x(t \to \infty)B(t \to \infty) = 0 \) and the asymptotic occupation numbers are defined only by the integral terms:

\[ n^{FC}_{\gamma}(t \to \infty) = \Gamma_{\gamma}^{FC}(t \to \infty) + \Gamma_{\gamma}(t \to \infty). \tag{28} \]

The following expressions are found for the asymptotes of these integral terms:

\[ \Gamma_{\gamma}^{FC}(t \to \infty) = \frac{g_0}{\pi} \int_0^\infty dw \frac{w^2 + \gamma^2}{\gamma^2 + w^2}\left(\frac{w^2 + \omega^2}{(w^2 + s_1^2)(w^2 + s_2^2)}\right), \tag{29} \]

and

\[ \Gamma_{\gamma}(t \to \infty) = \frac{g_0}{\pi} \int_0^\infty dw [1 + n_{\gamma}(w)]\left(\frac{w^2 + \gamma^2}{\gamma^2 + w^2}\right)\left(\frac{w^2 + \omega^2}{(w^2 + s_1^2)(w^2 + s_2^2)}\right). \tag{30} \]

For the FC fermionic oscillator, one can rewrite Eq. (28) as

\[ n^{FC}_{\gamma}(t \to \infty) = \Gamma_{\gamma}^{FC}(t \to \infty) + \Gamma_{\gamma}(t \to \infty). \tag{31} \]
to distinguish the term explicitly depending on temperature,
\[
I'_n = \frac{g_0}{\pi} \int_0^\infty dw n_s(w) = \frac{g_0 \gamma^2}{\pi} \left[ \frac{4w^2(w^2 + \gamma^2)}{\gamma^2 + w^2} \right] \ln \left( \frac{s_1}{s_2} \right) \left( s_1 - s_2 \right) \left( s_2 - s_1 \right)
\]
and the term independent of temperature,
\[
\Gamma_n = \frac{g_0 \gamma^2}{\pi} \left[ \frac{4w^2(w^2 + \gamma^2)}{\gamma^2 + w^2} \right] \ln \left( \frac{s_1}{s_2} \right) \left( s_1 - s_2 \right) \left( s_2 - s_1 \right)
+ g_0 \gamma^2 \left[ \frac{2(s_2 - \omega^2) \ln (s_2)}{2(s_2 - s_1)(s_2 - s_3)} + \frac{2(s_2 - s_1) \ln (s_2)}{2(s_2 - s_1)(s_2 - s_3)} \right]
+ g_0 \gamma^2 \omega \frac{(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)}{2(s_2 - s_1)(s_2 - s_3)(s_3 - s_1)}.
\]
By analogy the expression for the FC bosonic oscillator reads
\[
n_n^{\infty}(t \to \infty) = I'_n + \Gamma_n.
\]
\[\text{FIG. 6. The dependencies of asymptotic occupation numbers } n_n(t \to \infty) \text{ on the coupling strength } g_0 \text{ for the FC (solid lines) and RWA (dashed lines) fermionic oscillators. (a) The separate contributions of } I_n^c(t \to \infty) \text{ and } I_n^c(t \to \infty) \text{ in Eq. (28) are presented by dotted and dash-dotted lines, respectively. (b) The separate contributions of } I'_n \text{ and } \Gamma_n \text{ in Eq. (31) are presented by dotted and dash-dotted lines, respectively. The calculations are performed at } \gamma/\Omega = 12, \text{ and } kT/\hbar\Omega = 1.\]

where
\[
I'_n = \frac{g_0}{\pi} \int_0^\infty dw n_s(w) \times \frac{\gamma^2}{\gamma^2 + w^2} \left[ \frac{2w^2 + \gamma^2(w^2 + \omega^2)}{w^2 + s_1^2}(w^2 + s_2^2)(w^2 + s_3^2) \right],
\]
and the term \( I' \) is the same as in Eq. (31). These expressions are used further to study the asymptotic behavior of occupation numbers.

In Figs. 6 and 7, the dependencies of asymptotic occupation numbers \( n_n(t \to \infty) \) on the coupling strength \( g_0 \) are shown for fermionic and bosonic collective subsystems and for both types of couplings. In the case of RWA coupling (dashed lines), the values of \( n_n^{\text{RWA}}(t \to \infty) \) monotonically decrease with increasing \( g_0 \) for both fermionic and bosonic collective subsystem. The dependence of \( n_n^{\text{FC}}(t \to \infty) \) on \( g_0 \) is more complicated. For the fermionic collective subsystem, the value of \( n_n^{\text{FC}}(t \to \infty) \) falls down with increasing \( g_0 \) up to \( g_0 \approx 0.45 \) reaching some maximal value. The further increase of \( g_0 \) causes a decrease of \( n_n^{\text{FC}}(t \to \infty) \). Such a behavior of occupation number is due to the contributions of two terms, i.e., \( I_n^c(t \to \infty) \) [dotted line in Fig. 6(a)] and \( I'_n(t \to \infty) \) [dash-dotted line in Fig. 6(a)], which have different dependencies on \( g_0 \). As seen in Fig. 6(b), the temperature-dependent term (dotted line) decreases with increasing \( g_0 \) while the term \( I'_n \) (dash-dotted line) increases. So, the contributions of increasing and decreasing terms provide the complicated dependence of asymptotic occupation number on coupling strength. Note that the range of \( g_0 \) considered includes both weak- and strong-coupling regimes.
with these roots Eq. (33) reads

\[
\Gamma' = \frac{g_0}{\pi} \frac{\gamma^2(-\gamma^2 + \pi \gamma \omega - \omega^2 + \gamma^2 \ln \left(\frac{\pi}{\omega^2} \right) - \omega^2 \ln \left(\frac{\pi}{\omega^2} \right))}{(\gamma^2 + \omega^2)^2}.
\]

(37)

Assuming \(\omega \ll \gamma\), we get

\[
\Gamma' \approx g_0 \frac{1}{\pi} \ln \left(\frac{\gamma}{\omega} \right).
\]

(38)

Replacing the roots in Eqs. (32) and (35) by those from Eqs. (36), the temperature terms of asymptotic occupation numbers for fermionic and bosonic systems are

\[
\Gamma' = \frac{2g_0}{\pi} \left( \frac{kT}{\hbar \gamma} \right)^2 \left[ \zeta(2) + 6 \left( \frac{kT}{\hbar \gamma} \right)^2 \zeta(3) \right].
\]

(46)

for fermionic and bosonic systems, respectively. Here, \(\zeta(n)\) is the Riemann zeta function. Note that, in the case of a RWA oscillator we obtained in Ref. [41]

\[
\Gamma'_RWA = \frac{g_0}{2\pi} \left( \frac{kT}{\hbar \Omega} \right)^2 \left[ \zeta(2) + 4 \left( \frac{kT}{\hbar \Omega} \right)^2 \zeta(3) \right]
\]

(47)

for fermionic subsystem and

\[
\Gamma'_BOS = \frac{g_0}{\pi} \left( \frac{kT}{\hbar \Omega} \right)^2 \left[ \zeta(2) + 4 \left( \frac{kT}{\hbar \Omega} \right)^2 \zeta(3) \right]
\]

(48)

for the bosonic subsystem. While in the case of a RWA oscillator the ratio \(\Gamma'_RWA/\Gamma'_BOS \approx 1/2\), in the case of FC oscillator the ratio \(\Gamma'_F/\Gamma'_BOS\) depends on \(T\).

In Fig. 8, the temperature dependencies of asymptotic values of \(\Gamma'_{s}\) (solid lines) are shown for fermionic and bosonic collective subsystems. The calculations are performed at \(g_0 = 0.001, \gamma/\Omega = 12\). For fermionic (bosonic) system at temperatures \(kT/(\hbar \Omega) \gg 0.3\), \(\Gamma'_{F} (\Gamma'_{BOS})\) almost coincides with the usual Fermi–Dirac (Bose–Einstein) distribution. At low temperatures \(kT/(\hbar \Omega) \leq 0.04\) the expressions (45) and (46) seem to be a very good approximation. In Fig. 8, we also mark the value of a temperature-independent part \(\Gamma_F = 5.7 \times 10^{-4}\) of asymptotic occupation numbers. As seen, this term has a valuable contribution to the occupation number only at small temperature. Because at \(g_0 \to 0\) we get \(\Gamma' \to 0\) (see Figs. 6 and 7), the part \(\Gamma_F\) plays a major role in asymptotic occupation numbers at small temperature and relatively large \(g_0\).
The occupation numbers for the Fermi–Dirac and Bose–Einstein distributions are presented by dashed lines. The leading-order dependencies of asymptotic occupation numbers in FC fermionic and bosonic oscillators as well. The former term becomes important at low temperature and relatively large coupling strength. In FC oscillators, the contributions of these two terms provide a more complicated dependence of asymptotic occupation numbers on the coupling strength than in the case of a RWA oscillator. The transient time of occupation numbers is almost independent of the statistical nature of the bath. The results of numerical calculations of occupation numbers were found to be identical to those obtained with the discretized environment method.

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APPENDIX A

Using the general equations of motion for the creation $a^\dagger(t)$ and annihilation $a(t)$ operators (6) and the commutation relations $(a^\dagger)^2 = a^2 = 0$, we obtain the equation

$$\frac{dn_a(t)}{dt} = \frac{i}{\hbar} \sum_v g_v [a(t) - a^\dagger(t)][a^\dagger_v(t) - a_v(t)]$$

$$= \frac{i}{\hbar} \sum_v g_v [a^\dagger_v(t)a(t) - a^\dagger(t)a_v(t)]$$

$$+ a(t)a_v(t) - a^\dagger(t)a^\dagger_v(t)$$

(A1)

for the occupation number $n_a(t) = a^\dagger(t)a(t)$ of the collective subsystem. For the operators $a^\dagger_v(t)a(t) - a^\dagger(t)a_v(t)$ and $a(t)a_v(t) - a^\dagger(t)a^\dagger_v(t)$ in Eq. (A1), one can derive the following equations:

$$\frac{d^2}{dt^2}[a^\dagger_a(t) a(t) - a(t) a^\dagger_a(t)] = -[\omega + \omega_a(t)]^2[a^\dagger_a(t) a(t) - a(t) a^\dagger_a(t)]$$

$$+ \frac{2i g_\nu}{\hbar} \frac{d}{dt} (n_a - n_{a\nu})$$

$$\frac{d^2}{dt^2}(a(t) a^\dagger(t) - a^\dagger(t) a(t)) = -[\omega + \omega_a(t)]^2[a(t) a^\dagger(t) - a^\dagger(t) a(t)]$$

$$+ \frac{2i g_\nu}{\hbar} \frac{d}{dt} (n_a + n_{a\nu} - 1)$$

(A2)

where $n_{a\nu}(t) = a^\dagger(t)a(t)$ are the occupation numbers of the bosonic subsystem. To obtain Eqs. (A2), we employ the random-phase approximation for the heat-bath degrees of freedom:

$$\sum_\nu g_\nu (a^\dagger_a a_\nu + a_\nu a^\dagger_a) \simeq 2 g_\nu a^\dagger_a a_v$$

$$\sum_\nu g_\nu (a_\nu a^\dagger_a - a^\dagger_a a_\nu) \simeq 0$$

Substituting the formal solutions of Eqs. (A2) into Eq. (A1) and taking $a(0)a(0) = 0$, $a^\dagger(0)a^\dagger(0) = 0$, $a(0)a(0) = 0$, and $a^\dagger(0)a^\dagger(0) = 0$, we rewrite Eq. (A1) as

$$\frac{dn_a(t)}{dt} = \sum_\nu \int_0^t ds \{ \mathcal{W}_\nu(t-s) [n_a(s) - n_a(s)]$$

$$+ \mathcal{W}_\nu(t-s) [1 - n_a(s) - n_{a\nu}(s)] \}$$

(A3)
where

\[ \mathcal{W}^{-}_v = \frac{2g_v^2}{\hbar^2} \cos((\omega - \omega_v)[t - s]), \]

\[ \mathcal{W}^{\dagger}_v = \frac{2g_v^2}{\hbar^2} \cos((\omega + \omega_v)[t - s]). \]

This master equation is complemented by the set of master equations for \( n_a \):

\[ \frac{dn_a(t)}{dt} = \int_0^t ds \left\{ \mathcal{W}^{-}_v(t - s)[n_a(s) - n_a(s)] - \mathcal{W}^{\dagger}_v(t - s)[1 - n_a(s) - n_a(s)] \right\}. \quad (A4) \]

In the case of RW A coupling, \( \mathcal{W}^{\dagger}_v = 0 \) in Eqs. (A3) and (A4).

**APPENDIX B**

The Heisenberg equations of motion for the creation and annihilation operators of intrinsic subsystems are obtained by commuting corresponding operator with \( H \):

\[ \frac{d}{dt} a^\dagger_v(t) = i\omega_v a^\dagger_v + (1 - 2a^\dagger_v a_v) \frac{i\hbar}{\omega_v} g_v a^\dagger_v + a_v, \]

\[ \frac{d}{dt} a_v(t) = -i\omega_v a_v - (1 - 2a^\dagger_v a_v) \frac{i\hbar}{\omega_v} g_v a^\dagger_v + a_v. \quad (B1) \]

As in Eq. (6), we disregard here the terms proportional to \( 2a^\dagger_v a_v \) (see Appendix A). The solution of Eq. (B1) is

\[ a^\dagger_v(t) + a_v(t) = [e^{i\omega_v t} a^\dagger_v(0) + e^{-i\omega_v t} a_v(0)] + \frac{2g_v}{\hbar\omega_v} \left\{ -[a^\dagger_v(t) + a_v(t)] + \cos(\omega_v t)[a^\dagger_v(0) + a_v(0)] \right\} \]

\[ + \int_0^t \cos(\omega_v(t - \tau)) \frac{d}{d\tau}[a^\dagger_v(\tau) + a_v(\tau)]d\tau. \quad (B2) \]

Substituting Eq. (B2) into Eq. (6) and eliminating the bath variables from the equations of motion for the collective subsystem, we obtain a set of Langevin-type integro-differential stochastic dissipative equations:

\[ \frac{d}{dt}[a^\dagger(t) + a(t)] = i\omega_v[a^\dagger(t) - a(t)] - \frac{d}{dt}[a^\dagger(t) - a(t)] \]

\[ = \Omega[a^\dagger(t) + a(t)] + \int_0^\infty K(t - \tau) \frac{d}{d\tau}[a^\dagger(\tau) + a(\tau)]d\tau + F(t) + K(t)[a^\dagger(0) + a(0)], \quad (B3) \]

with the renormalized frequency

\[ \Omega = \omega - 2 \sum_v \frac{2g_v^2}{\hbar^2\omega_v}. \quad (B4) \]

dissipative kernel

\[ K(t - \tau) = 2 \sum_v \frac{2g_v^2}{\hbar^2\omega_v} \cos(\omega_v(t - \tau)), \quad (B5) \]

and random force

\[ F(t) = \sum_v F_v(t) = \frac{2}{\hbar} \sum_v g_v [e^{i\omega_v t} a^\dagger_v(0) + e^{-i\omega_v t} a_v(0)]. \quad (B6) \]

In Eqs. (B3), the operator \( F(t) \) plays a role of random force and depends on the initial conditions for the internal subsystem. The operators \( F_v(t) \) are usually identified in statistical physics with fluctuations because of the uncertainty in the initial conditions for heat-bath operators \( a^\dagger_v(0) \) and \( a_v(0) \). Note that, in the case of bosonic systems, the combinations \( a^\dagger(t) + a(t) \) and \( i[a(t) - a^\dagger(t)] \) of creation and annihilation operators are the analogy of real coordinates and momentum.

To find a solution of Eqs. (B3), we apply the Laplace transform:

\[ G(s) = \frac{1}{s}[-\omega R(s) + G(0)], \quad R(s) = \frac{1}{s}[R(0) + \Omega G(s) + sK(s)G(s) + F(s)]. \quad (B7) \]
where \( G(s) = \mathcal{L}[a^\dagger(t) + a(t)] \), \( R(s) = -i \mathcal{L}[a^\dagger(t) - a(t)] \), \( K(s) = \mathcal{L}[K(t)] \), and
\[
F(s) = \mathcal{L}[F(t)] = \frac{2}{\hbar} \sum_v g_v \left[ \frac{a_v^\dagger(0)}{s - i\omega_v} + \frac{a_v(0)}{s + i\omega_v} \right].
\]

Solving the set of algebraic equations (B7), we find
\[
G(s) = \frac{sG(0) - a_s[R(0) + F(s)]}{s^2 + s\omega + s\omega K(s)}, \quad R(s) = \frac{\Omega G(0) + s[R(0) + G(0)K(s) + F(s)]}{s^2 + s\omega + s\omega K(s)}.
\]

The Laplace transforms of \( a^\dagger(t) \) and \( a(t) \) are
\[
a^\dagger(s) = a^\dagger(0) + \frac{2s + i(\omega + \Omega) + is K(s)}{2[s^2 + s\omega + s\omega K(s)]} + \frac{i(\Omega - \omega) + is K(s)}{2[s^2 + s\omega + s\omega K(s)]} + \frac{i(s + i\omega)}{2[s + i\omega_v][s^2 + s\omega + s\omega K(s)]},
\]
\[
a(s) = a(0) + \frac{2s - i(\omega + \Omega) - is K(s)}{2[s^2 + s\omega + s\omega K(s)]} + \frac{2s - i(\omega + \Omega) - is K(s)}{2[s^2 + s\omega + s\omega K(s)]} + \frac{-i(s - i\omega)}{2[s - i\omega_v][s^2 + s\omega + s\omega K(s)]}.
\]

The explicit solutions for the originals are
\[
a^\dagger(t) = a^\dagger(0)A^*(t) + a(0)B(t) + \sum_v \frac{g_v}{\hbar} a_v^\dagger(0)M_v^*(t, w_v) + \sum_v a_v(0)\frac{g_v}{\hbar} N_v(t, w_v),
\]
\[
a(t) = a(0)A(t) + a^\dagger(0)B^*(t) + \sum_v \frac{g_v}{\hbar} a_v^\dagger(0)N_v^*(t, w_v) + \sum_v a_v(0)\frac{g_v}{\hbar} M_v(t, w_v),
\]
where the time-dependent coefficients are denoted as follows:
\[
A(t) = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2s + i(\omega + \Omega) - is K(s)}{2[s^2 + s\omega + s\omega K(s)]} \right\},
\]
\[
B(t) = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{i(\Omega - \omega) + is K(s)}{2[s^2 + s\omega + s\omega K(s)]} \right\},
\]
\[
M(t, w_v) = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{-i(s - i\omega)}{2(s - i\omega_v)[s^2 + s\omega + s\omega K(s)]} \right\},
\]
\[
N(t, w_v) = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{i(s + i\omega)}{2(s - i\omega_v)[s^2 + s\omega + s\omega K(s)]} \right\}.
\]

So, one can write the time-dependent operator of the occupation number:
\[
a^\dagger(t)a(t) = a^\dagger(0)a(0)A^*(t)A(t) + a(0)a^\dagger(0)B^*(t)B(t) + \sum_v \frac{g_v^2}{\hbar^2} a_v^\dagger(0)a_v(0)M_v^*(t, w_v)M(t, w_v) + \sum_v \frac{g_v^2}{\hbar^2} a_v(0)a_v^\dagger(0)N_v^*(t, w_v)N(t, w_v).
\]

To find the explicit expressions for the occupation number, it is convenient to introduce the spectral density \( \rho(w) \) of the heat-bath excitations, which allows us to replace the sum over different two-level systems \( v \) by the integral over the frequency: \( \sum_v \longrightarrow \int_0^\infty dw \rho(w) \cdot \cdot \cdot \). Let us consider the following spectral function [2]:
\[
\frac{\rho(w)w^2}{\hbar^2} = \frac{1}{\pi} g_0 \frac{\gamma^2}{\gamma^2 + w^2},
\]
where the memory time \( \gamma^{-1} \) of the dissipation is the inverse of the bandwidth of the heat-bath excitations which are coupled to the collective system. This is the Ohmic dissipation with the Lorentzian cutoff (D rude dissipation). The relaxation time of the heat bath should be much less than the characteristic collective time; i.e., \( \gamma \gg \omega \). Employing Eq. (B11), we obtain the following expressions for the dissipative kernel:
\[
K(t) = \frac{g_0 \gamma^2}{\pi} \int_0^\infty dw \cos(wt) \frac{\cos(wt)}{\gamma^2 + w^2} = 2g_0 e^{-\gamma t},
\]
where \( g_0 = \frac{\hbar^2}{\pi \gamma^2} \), \( \gamma = \frac{1}{\hbar} \).
and
\[ K(s) = \frac{2g_{0}^2}{s + \gamma}. \]  

(B13)

This type of spectral function and dissipative kernel leads to expressions (7)–(9) for the occupation number.

The fluctuation-dissipation relations connect the dissipation of a collective subsystem and the fluctuations of random forces. These relations express the nonequilibrium behavior of the system in terms of equilibrium or quasi-equilibrium characteristics. They ensure that the system approaches the equilibrium state. We consider the initial distribution of bath fermionic operators \( a'_0(0) \) and \( a_0(0) \). For the correlation of the random force one can obtain
\[ \langle \langle F_{\nu}(t) \rangle \rangle = \langle \langle F_{\nu}(t) F_{\mu}(t) \rangle \rangle = 0, \]
and
\[ \langle \langle F_{\nu}(t) F_{\nu}(\tau) \rangle \rangle = \frac{4g_{0}^2}{\hbar^2} \left[ e^{i\omega_{\nu}(t-\tau)} \langle \langle a'_{\nu}(0) a_{\nu}(0) \rangle \rangle + e^{-i\omega_{\nu}(t-\tau)} \langle \langle a_{\nu}(0) a'_{\nu}(0) \rangle \rangle \right]. \]

Using these expressions, one can get the fluctuation dissipation relations
\[ K(t-\tau) = \frac{1}{2} \sum_{\nu} \langle \langle F_{\nu}(t) F_{\nu}(\tau) + F_{\nu}(\tau) F_{\nu}(t) \rangle \rangle. \]  

(B14)

Here, the symbol \( \langle \cdot \cdot \cdot \rangle \) denotes the average over the bath. Fulfillment of the fluctuation-dissipation relations means that we have correctly defined the dissipative kernels in the non-Markovian equations of motion.