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VECTOR FIELDS OF ZERO TOTAL CURVATURE OF THE SECOND KIND IN FOUR-DIMENSION SPACE

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Geometry of flat vector fields for which total curvature of the second kind equals zero in a domain of four-dimensional Euclidean space has been studied. These vector fields are classified depending on rank of the fundamental linear operator. Geometrical properties of non-holonomic Pfaffian variety orthogonal to the vector field are investigated for each class. An example of a vector field with constant nonholonomicity vector different from zero is constructed. The research is carried out by the Cartan's method of exterior forms within moving frames.

Introduction

Let $\boldsymbol{v}$ be the flat vector field without particular points in a $G\subset\boldsymbol{E}_4$ domain. Connect the field $\boldsymbol{v}$ to moving orthonormal frame $\{M;\vec{e}_\alpha\}$, $M\in G$. Its derivative formula is written down in the form

$$d\vec{r}=\omega^\alpha\vec{e}_\alpha,\, d\vec{e}_\alpha=\omega_\alpha^\beta\vec{e}_\beta,$$

where $\vec{r}$ is radius-vector of point M,

$$\vec{e}_4=\frac{v}{|\vec{v}|},\omega_\alpha^\beta=-\omega_\beta^\alpha,d\omega^\alpha=\omega^\beta\wedge\omega_\beta^\alpha,d\omega_\alpha^\beta=$$

$$=\omega_\alpha^\gamma\wedge\omega_\gamma^\beta,(\alpha,\beta,\gamma=\overline{1,4}).$$

In this case the Pfaffian equation $\omega^4=0$ set three-dimensional distribution (M,π_3) [1], i. e. flat map, correlating any point $M\in\boldsymbol{G}$ with hyperplane π_3, orthogonal to the field vector $\boldsymbol{v}$ in this point. All integral lines and surfaces of the equation $\omega^4=0$, crossing M, are tangent to the hyperplane π_3 in the given point. Let us name a system of all integral curves and surfaces of the Pfaffian equation $\omega^4=0$ (following [2]) a Pfaffian variety, orthogonal to the vector field $\boldsymbol{v}$. The plane π_3 is the tangential plane of the Pfaffian variety $\omega^4=0$ in the point M. If the equation $\omega^4=0$ is completely integrated, then one integral hyperplane goes through each point M, and we have a foliation [3]. In this case the Pfaffian variety is called holonomic [1, 2]. In the opposite case it is non-holonomic. We shall study the vector field with non-holonomic Pfaffian variety.

The fundamental invariants of the vector field (as well as the Pfaffian variety orthogonal to it) coincide with invariants of the fundamental linear operator A [4], determined by the formula

$$A(d\vec{r})=d\vec{e}_4.$$

Its matrix in the basis $\{\vec{e}_\alpha\}$ coincides with the matrix (A_β^α), obtained in decomposition of the main Pfaffian forms ω_4^α in terms of basis forms ω^α:

$$\omega_4^\alpha=A_\beta^\alpha\omega^\beta. \quad (1)$$

One of the characteristic values λ_0 of the operator A is equal to zero. The latent vectors $\vec{\xi}$ (ξ^α) corresponding to $\lambda_0=0$, are defined by the set of equations

$$A_\alpha^i\xi^\alpha=0,(i=1,2,3) \quad (2)$$

and are tangent to equidirectional lines (lines, along which the field vectors are parallel [2, 4]). Depending on rang of the operator A through the point M either one equidirectional line (rang A=3) or equidirectional plane – two-dimension (rang A=2) or three-dimension (rang A=1) goes through the point M. All these cases we study below.

In the paper the following designations are used: $k_i^{(2)}$ are the principal curvatures of the 2-nd kind, $K_2=-k_1^{(2)}k_2^{(2)}k_3^{(2)}$ is the total curvature of the 2-nd kind, $\vec{\rho}=\rho^i\vec{e}_i$ is the non-holonomicity vector, are the principal directions of the 1-st kind, $K_1=-k_1^{(1)}k_2^{(1)}k_3^{(1)}$ is the total curvature of the 1-st kind, A^* is the contraction of A operator onto the plane π_3 [4].

Vector field, for which K_2=0 and rang A=3

Such a vector field is characterized by the fact that through every point M goes the only equidirectional line, this line belonging to the Pfaffian variety $\omega^4=0$. Direct the vector $\vec{e}_1$ at a tangent to the equidirectional line, then from (2) it follows that $A_1^1=A_1^2=A_1^3=0$. But as rang A=3, then the determinant

$$\Delta=\begin{vmatrix}A_2^1 & A_3^1 & A_4^1\\ A_2^2 & A_3^2 & A_4^2\\ A_2^3 & A_3^3 & A_4^3\end{vmatrix}\neq 0.$$

Let us show that for the investigated field (K_2=0, rang A=3) equidirectional line is also the line of curvature of the 2-nd kind. For this purpose it is enough to show that one of the principal directions of the 2-nd kind coincides with the vector direction $\vec{e}_1$. The principal curvatures of the 2-nd kind [4] are different only in sign from the equation roots

$$\begin{vmatrix} A_1^1-\lambda & A_2^1 & A_3^1 \\ A_1^2 & A_2^2-\lambda & A_3^2 \\ A_1^3 & A_2^3 & A_3^3-\lambda \end{vmatrix} = 0.$$

Since K_2=0, then at least one of the principal curvatures of the 2-nd kind equals to zero, let $k_1^{(2)}=0$. Corresponding latent vectors, determining the principal direction of the 2-nd kind, is found from the set of equations

$$A_j^i \xi^j = 0, (i, j = 1,2,3),$$

but as $A_1^1=A_1^2=A_1^3=0$, then in the plane π_3 we obtain the vector, for which $\xi^2=\xi^3=0$, i. e. The vector $\vec{e}_1$ is a vector of the principal direction of the 2-nd kind.

Show that equidirectional line is in addition an asymptotic line of the Pfaffian variety ω^4=0. Find the asymptotic lines, for them

$$\left\langle d^2\vec{r}, \vec{e}_1, \vec{e}_2, \vec{e}_3 \right\rangle = 0.$$

Hence, using derivation formula, we arrive at the equation

$$A_2^2(\omega^2)^2 + A_3^3(\omega^3)^2 + A_2^1\omega^1\omega^2 + $$
$$+A_3^1\omega^1\omega^3 + (A_3^2 + A_2^3)\omega^2\omega^3 = 0, \qquad (3)$$

defining the asymptotic lines at ω^4=0. The equation (3) at $\omega^2=\omega^3=\omega^4=0$ is turned into identity. Since $\omega^2=\omega^3=\omega^4=0$ are the equidirectional lines, it means that any equidirectional lines is an asymptotic line. The asymptotic line coinciding with the equidirectional one lies in the plane π_3. Tangents to the asymptotic lines, going through the point M, form a cone

$$A_2^2(x^2)^2 + A_3^3(x^3)^2 + A_2^1x^1x^2 + $$
$$+A_3^1x^1x^3 + (A_3^2 + A_2^3)x^2x^3 = 0. \qquad (4)$$

Let us investigate the set of planes π_3. Find the plane characteristic π_3 when displaced in any curve, going through the point M:

$$x^4 = 0,$$
$$(A_2^1\omega^2 + A_3^1\omega^3 + A_4^1\omega^4)x^1 + (A_2^2\omega^2 + A_3^2\omega^3 + A_4^2\omega^4)x^2 + $$
$$+(A_2^3\omega^2 + A_3^3\omega^3 + A_4^3\omega^4)x^3 - \omega^4 = 0. \qquad (5)$$

The equation (5) includes only three basic formulas. It means that the set of planes π_3 depends on just three parameters (but not on four as it is in general case). Characteristic point M_0 of the plane π_3 has the coordinates

$$M_0\left(\frac{\begin{vmatrix} A_2^2 & A_2^3 \\ A_3^2 & A_3^3 \end{vmatrix}}{\Delta}, \frac{\begin{vmatrix} A_2^3 & A_2^1 \\ A_3^3 & A_3^1 \end{vmatrix}}{\Delta}, \frac{\begin{vmatrix} A_2^1 & A_2^2 \\ A_3^1 & A_3^1 \end{vmatrix}}{\Delta}, 0 \right), \Delta \neq 0.$$

That is the planes π_3 have a envelope – three-dimensional surface, described by the points M_0. Having substituted the coordinates of the point M_0 in the equation (4), we ascertain that the point M_0 lies on the tangent to one of asymptotic lines, going through the point M. Hence, the line MM_0 is the tangent to one of the asymptotic lines.

Place the vector $\vec{e}_2$ on the plane $\{M, \vec{e}_1, \overline{MM_0}\}$, then $A_2^1A_3^2=A_3^1A_2^2$ and the frame becomes canonical. Note that characteristics of the plane π_3 at displacement in curves, belonging to ω^4=0, form a bundle, axis of which is the line MM_0, (it follows from (5)).

Let us find the principal directions of the 2-nd kind corresponding to the curvatures $k_2^{(2)}, k_3^{(2)}$. From the equation

$$\lambda^2 - (A_2^2 + A_3^3)\lambda + A_2^2A_3^3 - A_3^2A_2^3 = 0$$

we find $\lambda^2=-k_2^{(2)}$, $\lambda^3=-k_3^{(2)}$. Having calculated the principal directions of the 2-nd kind corresponding to the curvatures $k_2^{(2)}, k_3^{(2)}$, we see that they are orthogonal to the line MM_0. Thus, the following theorem has been proved.

Theorem 1. *Smooth vector field, for which* K_2=0 *and* rang A=3 *possesses the following properties*: 1) *only one of the principal curvatures of the* 2*-nd kind is equal to zero* ($k_1^{(2)}=0$); 2) *one equidirectional line goes through the point* M, it coincides *with the curvature line of the* 2*-nd kind and is an asymptotic line lying in the plane* π_3, 3) *a set of planes* π_3 *depends on the three parameters and has a three-dimensional surface as an envelope*; 4) *the plane characteristics* π_3, *obtained at displacement in all curves belonging to* ω^4=0, *form a bundle with the axis* MM_0 (M_0 *is the point of envelope in the plane* π_3); 5) *in the point M the principal directions of the* 2*-nd kind, corresponding to* $k_2^{(2)}$ *and* $k_3^{(2)}$, *are orthogonal to the line* MM_0; 6) *the line* MM_0 *is a tangent to the asymptotic line not lying in the plane* π_3.

Vector fields, for which K_2=0 and rang A=2

In this case one 2-dimentional equidirectional surface goes through every point $M \in G$, the surface being an integral surface of the Pfaffian set of equation:

$$A_1^1\omega^1 + A_2^1\omega^2 + A_3^1\omega^3 + A_4^1\omega^4 = 0,$$
$$A_1^2\omega^1 + A_2^2\omega^2 + A_3^2\omega^3 + A_4^2\omega^4 = 0. \qquad (6)$$

Denote

$$A_* = \begin{pmatrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^2 & A_2^2 & A_3^2 \end{pmatrix}.$$

The tangent plane $\boldsymbol{T}_2$ of equidirectional surface in the point M is determined by the equations

$$A_1^1x^1 + A_2^1x^2 + A_3^1x^3 + A_4^1x^4 = 0,$$
$$A_1^2x^1 + A_2^2x^2 + A_3^2x^3 + A_4^2x^4 = 0. \qquad (7)$$

The plane $\boldsymbol{T}_2$ either crosses the plane π_3 along the line, when rang A_*=2, or belongs to the plane π_3, when rang A_*=1.

Consider each of these possibilities.

a) Let rang A_*=2. Direct the vector $\vec{e}_1$ along the intersection line of $\boldsymbol{T}_2$ and π_3, then

$$A_1^1 = A_1^2 = 0, \begin{vmatrix} A_2^1 & A_3^1 \\ A_2^2 & A_3^2 \end{vmatrix} \neq 0.$$

As K_2=0, then A_1^3=0. It is easy to check that in the point M one of the curvature lines of the 2-nd kind (that corresponds to the curvature $k_1^{(2)}=0$) lies on equidirectional surface, i.e. it is a equidirectional line. Besides, this line lies in the plane π_3 and represents an asymptotic line. The plane set π_3 depends on the three parameters, but due to the fact that rang $A=2$, this set does not have an envelope. Characteristics of the plane π_3, obtained at displacement along the curves from $\omega^4=0$, are the two-dimensional planes, crossing one line

$$A_2^1x^1 + A_2^2x^2 + A_2^3x^3 = 0,$$
$$A_3^1x^1 + A_3^2x^2 + A_3^3x^3 = 0,$$
$$x^4 = 0. \qquad (8)$$

But characteristics of the plane π_3, obtained at displacement along the curve not belonging to $\omega^4=0$, are parallel to the line. The line (8) is tangent to some asymptotic line, not lying in the plane π_3. One can show that the principal directions of the 2-nd kind corresponding to the curvatures $k_2^{(2)}\neq0$, $k_3^{(2)}\neq0$ are orthogonal to this line. By doing this the following theorem has been proved.

Theorem 2. *Smooth vector field, for which* $K_2=0$, rang$A=2$, rang $A_*=2$ possesses *the following properties*: 1) *only one of the principal curvatures of the* 2*-nd kind equals to zero* ; 2) *one* 2*-dimensional equidirectional surface, on which one of lines is a line of the curvature of the second kind* (*that which satisfies the curvature* $k_1^{(2)}=0$) *goes through every point M, it is an asymptotic line as well, lying in the plane* π_3; 3) *a set of planes* π_3 *depends on the three parameters, but it does not have an envelope*; 4*) the plane characteristics* π_3, *obtained at displacement in the curves from* $\omega^4=0$ *form a bunch, the axis of which is tangent to the asymptotic line, not coinciding with the curvature line of the* 2*-nd kind*; 5) *the curvature line of the* 2*-nd kind, corresponding to the curvatures* $k_2^{(2)},k_3^{(2)}$, *are orthogonal to that asymptotic one*; 6) *the plane characteristics* π_4, *obtained at displacement in the curves, not belonging to* $\omega^4=0$, *are parallel to the bunch axis.*

b) Let rang $A_*=1$. In this case in the point $M\in G$ the plane $T_2\subset\pi_3$ and 2-dimensional equidirectional surface is an integral surface for $\omega^4=0$.

Place the vectors $\vec{e}_1,\vec{e}_2$ on the plane T_2, then we obtain

$$A_1^1 = A_2^1 = A_1^2 = A_2^2 = 0, \begin{vmatrix} A_3^1 & A_4^1 \\ A_3^2 & A_4^2 \end{vmatrix} \neq 0.$$

Besides, as rang $A=2$, то $A_1^3=A_2^3=0$, $(A_3^1)^2+(A_3^2)^2\neq0$. The equidirectional surfaces are defined by the equations after that

$$\omega^3 = \omega^4 = 0, \qquad (9)$$

but the asymptotic lines – by the equations

$$(A_3^1\omega^1 + A_3^2\omega^2 + A_3^3\omega^3)\omega^3 = 0, \omega^4 = 0.$$

Thus, set of all asymptotics breaks up into two sets, one of which coincides with (9), hence, is holonomic, the second one is non-holonomic as a set of equations

$$A_3^1\omega^1 + A_3^2\omega^2 + A_3^3\omega^3 = 0, \omega^4 = 0 \qquad (10)$$

is not completely integrated.

The cone of tangents to asymptotic lines in the point M breaks up into two two-dimensional planes. One of them coincides with T_2, the second T_2^* has the equation

$$A_3^1x^1 + A_3^2x^2 + A_3^3x^3 = 0, x^4 = 0.$$

having directed the vector $\vec{e}_1$ along the intersection line of the planes T_2 and T_2^*, we obtain $A_3^1=0, A_3^2\neq0$. $A_3^3=-k_3^{(2)}=H$, $k_1^{(2)}=k_2^{(2)}=0$, $A_3^2=k_3^{(2)}\mathrm{tg}\varphi$, where φ is the angle between the planes T_2 and T_2^*, if $k_3^{(2)}\neq0$ If $k_3^{(2)}=0$, the plane T_2^* is orthogonal to the plane T_2.

The curvature vector $k\vec{n}$ line of flow of the vector field is determined by the formula

$$k\vec{n} = A_4^1\vec{e_1} + A_4^2\vec{e_2} + A_4^3\vec{e_3}.$$

Any direction of the plane T_2 is a principal direction of the 2-nd kind corresponding to the curvature $k_1^{(2)}=k_2^{(2)}=0$. In this case, if $A_3^3=0$, then $k_3^{(2)}=0$ and other directions of the 2-nd kind in the point M does not exist. If $A_3^3\neq0$, then to the curvature $k_3^{(2)}=-A_3^3$ corresponds to the principal direction $A_3^2\vec{e}_2+A_3^3\vec{e}_3$, orthogonal to the plane T_2^*. The curvature lines of the 2-nd kind corresponding $k_1^{(2)}=k_2^{(2)}=0$, coincide with equidirectional lines, i. e. in each point M they form 2-dimensional surface – integral for the equation $\omega^4=0$. At $A_3^3=0$ there are not any other curvature lines of the 2-nd kind, but at $A_3^3\neq0$ one more curvature line of the 2-nd kind not belonging to equidirectional surface goes through the point M. This line is defined by a set of equation

$$A_3^3\omega^2 - A_3^2\omega^3 = 0, \omega^1 = \omega^4 = 0.$$

Calculating the principal curvatures and principal directions of the 1-st kind we see that one of the principal curvatures of the 1-st kind is equal to zero ($k_1^{(1)}=0$) and the principal direction coinciding with the vector direction $\vec{e}_1$ corresponds to it. Thus, for the given class of the vector fields not only $K_2=0$, but also $K_1=0$. However $\omega^4=0$ remains non-holonomic variety, since the vector of non-holonomicity $\vec{\rho}=1/2A_3^2\vec{e}_1\neq\vec{0}$. The direction of non-holonomicity vector coincide with the principal direction of the 1-st kind (note that it is also one of the principal directions of the 2-nd kind). If $A_3^3=0$ ($k_3^{(2)}=0$), then the principal direction of the 1-st kind are also the directions (0: 1: ±1), coinciding with the directions of angle bisector between the lines obtained in the plane section T_2 and T_2^* by the plane orthogonal to the line $T_2\cap T_2^*$.

Consider the set of planes π_3. Find the plane characteristics obtained at its displacement in any direction:

$$x^4 = 0,$$
$$A_4^1\omega^4x^1 + (A_3^2\omega^3 + A_4^2\omega^4)x^2 + (A_3^3\omega^3 + A_4^3\omega^4)x^3 + \omega^4 = 0.$$

It is seen from here: 1) a set of planes π_3 depends only on two parameters; 2) all characteristics of the plane π_3 form a bunch with axis

$$x^4 = 0,$$
$$A_3^2x^2 + A_3^3x^3 = 0,$$
$$A_4^1x^1 + A_4^2x^2 + A_4^3x^3 = 0;$$

3) at displacement in any curve from $\omega^4=0$, not lying on equidirectional surface we have one and the same characteristic of the plane π_3, coinciding with the plane T_2^*.

As a result the following statement has been proved.

Theorem 3. *Smooth vector field, for which* $K_2=0$, rang $A=2$, rang $A^*=1$ *possesses the following properties*: 1) *at le-*

ast two of the principal curvatures of the 2*-nd kind are equal to zero* ($k_1^{(2)}=k_2^{(2)}=0$); 2) 2*-dimensional equidirectional surface being an integral surface for* $\omega^4=0$ *and belonging to the hyperplane* π_3 *goes through any point M*; 3) *all lines of equidirectional surface are curvature lines of the 2-nd kind corresponding to the curvatures* $k_1^{(2)}=k_2^{(2)}=0$, *as well as to asymptotic lines*; 4) *cone of tangents to asymptotic lines in the point M breaks up into two*, *intersecting along the line*, *two-dimensional planes* T_2 *and* T_2^*, *one of which* (T_2) *is a tangent plane to equidirectional surface*; 5) *a set of all asymptotic lines breaks up into two sets*: *holonomic*, *fibering into equidirectional surface, and nonholonomic one with tangent planes* T_2^* *at every point M*; 6) *total curvature* K_1 *of the first kind is also equal to zero*, *but among the principal curvatures of the* 1*-st kind only one vanishes* ($k_1^{(1)}=0, k_2^{(1)}\neq 0, k_3^{(1)}\neq 0$); 7) *direction of the line* $T_2\cap T_2^*$ *is a principal direction of the* 1*-st kind corresponding to* $k_1^{(1)}=0$; 8) *at* $k_1^{(2)}=k_2^{(2)}=k_3^{(2)}=0$ *all directions of the plane* T_2 *are principal directions of the* 2*-nd kind*, *there are no other principal directions of the* 2*-nd kind*; 9) *at* $k_1^{(2)}=k_2^{(2)}=0, k_3^{(2)}\neq 0$ *besides principal directions of the* 2*-nd kind lying in* T_2, *there is one more principal direction of the* 2*-nd kind*, *orthogonal to* T_2^* *and the line* $T_2\cap T_2^*$; 10) *a set of planes* π_3 *depends on only two parameters and has a tree-dimensional a torse with rectilinear generator as an envelope*; 11) *the plane characteristics* π_3, *obtained at displacement in any curve from* $\omega^4=0$, *is the plane* T_2^*.

Vector fields, for which $K_2=0$ and rang $A=1$

If rang $A=1$, then through every point $M\in G$ goes one three-dimensional equidirectional surface [3]. The tangent plane T_3 of this surface either cross the plane π_3 along the two-dimensional plane $T_2=\pi_3\cap T_3$, or $T_3=\pi_3$. In the latter case the Pfaffian variety $\omega^4=0$ is holonomic and we have a foliation [3], the fibers of which are three-dimensional ruled surfaces. We leave aside this case and pass on the consideration of the first one.

The plane $T_2=\pi_3\cap T_3$ is defined by the equations

$$A_1^2x^1 + A_2^2x^2 + A_3^2x^3 = 0,$$
$$x^4 = 0. \quad (11)$$

Place the vectors $\vec{e}_1,\vec{e}_2$ on this plane, then $A_1^2=A_2^2=0$, $A_3^2\neq 0$ and the system (11) has the view

$$x^3 = x^4 = 0. \quad (12)$$

As in the case considered rang $A=1$, then

$$A_1^1 = A_2^1 = A_1^3 = A_2^3 = 0, \begin{vmatrix} A_3^1 & A_4^1 \\ A_3^2 & A_4^2 \end{vmatrix} = 0, \begin{vmatrix} A_3^2 & A_4^2 \\ A_3^3 & A_4^3 \end{vmatrix} = 0. \quad (13)$$

Under these conditions the cone of tangents to asymptotic lines in the point M breaks up into two two-dimensional planes: T_2 and T_2^*. The plane T_2 has the equations (12), the plane T_2^* – the equations

$$A_3^1x^1 + A_3^2x^2 + A_3^3x^3 = 0,$$
$$x^4 = 0.$$

Since $A_3^2\neq 0$, one can put $A_3^1=0$ directing the vector $\vec{e}_1$ along the line $T_2\cap T_2^*$. Besides, from (13) it follows that $A_4^1=0$. After that the frame $\{M,\vec{e}_\alpha\}$ becomes canonical. In it the line curvature vector of the field flow $\boldsymbol{v}$ is determined by the formula

$$k\vec{n} = A_4^2\vec{e}_2 + A_4^3\vec{e}_3.$$

I.e. osculating plane of the field flow line $\boldsymbol{v}$ is orthogonal to the line $T_2\cap T_2^*$, but nonholonomicity vector

$$\vec{\rho} = \tfrac{1}{2}A_3^2\vec{e}_1 \neq 0$$

is parallel to this line.

Find the principal curvatures and the principal directions of the 2-nd kind; obtain $k_1^{(2)}=k_2^{(2)}=0, k_3^{(2)}=-A_3^3$. The principal directions of the 2-nd kind in the point M, corresponding to the curvatures $k_1^{(2)}=k_2^{(2)}=0$, are all directions of the plane T_2. But the direction corresponding to $k_3^{(2)}=-A_3^3$, is the vector direction $A_3^2\vec{e}_2+A_3^3\vec{e}_3$, which at $A_3^3=0$ lies in the plane T_2, but at $A_3^3\neq 0$ is orthogonal to the plane T_2^*.

All asymptotic lines tangent to T_2, lie in the tree-dimensional plane π_3 and coincide with those curvature lines of the second kind, which correspond to the curvatures $k_1^{(2)}=k_2^{(2)}=0$.

For the principal curvatures of the 1-st kind we find the formulas

$$k_1^{(1)} = 0, k_{2,3}^{(1)} = \tfrac{1}{2}(-k_3^{(2)} \pm \sqrt{(k_3^{(2)})^2 + 4(\rho^1)^2}).$$

The principal direction of the 1-st kind, corresponding to the curvature $k_1^{(1)}=0$, coincides with the direction of the line $T_2\cap T_2^*$. This is the case when $K_1=K_2=0$, but the Pfaffian variety, orthogonal to the vector field, remains nonholonomic, even if principal curvatures of the 2-nd kind are zeroes. Note that at $k_1^{(2)}=k_2^{(2)}=k_3^{(2)}=0$ the planes T_1 and T_2^* are orthogonal.

For the given class of vector fields the characteristics of the plane π_3 are defined by the equations

$$x^4 = 0,$$
$$(A_3^2\omega^3 + A_4^2\omega^4)x^2 + (A_3^3\omega^3 + A_4^3\omega^4)x^3 - \omega^4 = 0.$$

It is thus seen that a set of all planes π_3 depends only on two parameters. At displacement in any curve from $\omega^4=0$, not lying on equidirectional surface, we have the same characteristic – the plane T_2^*. At displacement in curves, not lying in $\omega^4=0$, characteristic of the plane π_3 are parallel to T_2^*. Thus, two-parametric family of planes π_3 does not have an envelope and consists of tangent planes of one-parametric family of torses with two-dimensional flat generators. Thus, we have arrived at the following statement.

Theorem 4. *Smooth vector field*, *for which* $K_2=0$, *and the nonholonomicity vector* $\vec{\rho}\neq 0$ *possesses the following properties*: 1) *At least two principal curvatures of the second kind are equal to zero* ($k_1^{(2)}=k_2^{(2)}=0$); 2) *through every point M goes one* 3*-dimensional equidirectional surface, the tangent plane* T_3 *of which does not cross the plane* π_2 *in two-dimensional plane* T_2; 3) *the curvature lines of the* 2*-nd kind, corresponding to the curvatures* $k_1^{(2)}=k_2^{(2)}=0$, *form 2-dimensional surface in the point M, it lies in the plane* π_3 *and has a tangent plane coinciding with* T_2; 4) *cone of tangents to asymptotic lines in the point breaks up into a couple of two-dimensional planes* T_2 *and* T_2^*, *not intersecting in the line, in this case asymptotic lines tangent to* T_2, *coincide with the* *curvature lines of the* 2*-nd kind*; 5) *one of the curvature lines of the* 1*-st kind equals to zero* ($k_1^{(1)}$=0), *but the principal direction of the* 1*-st kind corresponding to it coincides with the direction of the line* $T_2 \cap T_2^*$; 6) *the planes* T_2 *and* T_2^* *are orthogonal only at* $k_1^{(2)}=k_2^{(2)}=k_3^{(2)}=0$, *in this case all principal curvatures of the* 2*-nd kind, crossing the point M, belong to two-dimensional surface;* 7) *if* $k_3^{(2)}\neq 0$, *the principal direction of the 2-nd kind corresponding to the curvature* $k_3^{(2)}\neq 0$, *is orthogonal to the plane* T_2^* *and the line* $T_2 \cap T_2^*$; 8) *a set of planes* π_3, *orthogonal to the vector field, depends on two parameters, does not have an envelope and consists of tangent planes of one-dimensional family of torses with* 2*-dimensional flat generators.*

Theorem 5. *There is the only vector field of the class* $k_1^{(2)}=k_2^{(2)}=k_3^{(2)}=0$ *with straight lines of flow and nonholonomicity vector constants not equal to zero.*

Proof. Let $k_1^{(2)}=k_2^{(2)}=k_3^{(2)}=0$ and flow lines of the vector field are straight lines. Then $A_4^1=A_4^2=A_4^3=0$, the formulas (1) take on the form

$$\omega_4^1 = 0,$$
$$\omega_4^2 = 2\rho^1\omega^3,$$
$$\omega_4^3 = 0. \qquad (14)$$

Nonholonomicity vector $\vec{\rho}$ in this case is determined by the formula

$$\vec{\rho} = \rho^1 \vec{e}_1.$$

Necessitate the vector $\vec{\rho}$ to be constant, not equal to zero vector. Then

$$d\vec{\rho} = \vec{0}.$$

I. e. $d\rho\vec{e}_1+\rho^1(\omega_1^2\vec{e}_2+\omega_1^3\vec{e}_3)=\vec{0}$. Hence,

$$\rho^1 = \text{const} \neq 0, \omega_1^2 = \omega_1^3 = 0. \qquad (15)$$

In terms of (14), (15) for exterior differentials of base forms we have

$$d\omega^1 = 0, d\omega^2 = 2\rho^1\omega^4 \wedge \omega^3, d\omega^3 =$$
$$= \omega^2 \wedge \omega_2^3, d\omega^4 = 2\rho^1\omega^3 \wedge \omega^2.$$

Differentiating the forms in the exterior way (14) and applying then the Cartan's lemma, we obtain $\omega_2^3=0$ and then $d\omega^3=0$.

As $d\omega^1=0$, $d\omega^3=0$, one can put $\omega^1=dt$, $\omega^3=dy$. Besides, denote $2\rho^1=\alpha$. After that derivation formulas of the frame have the view

$$d\vec{r} = dt\vec{e}_1 + \omega^2\vec{e}_2 + dy\vec{e}_3 + \omega^4\vec{e}_4,$$
$$d\vec{e}_1 = \vec{0},$$
$$d\vec{e}_2 = -\alpha dy\vec{e}_4,$$
$$d\vec{e}_3 = \vec{0},$$
$$d\vec{e}_4 = \alpha dy\vec{e}_2. \qquad (16)$$

From (16) we have

$$\vec{e}_1 = \vec{\varepsilon}_1, \vec{e}_3 = \vec{\varepsilon}_3, \frac{d\vec{e}_2}{dy} = -\alpha\vec{e}_4, \frac{d^2\vec{e}_2}{dy^2} = -\alpha\vec{e}_2.$$

Consequently,

$$\vec{e}_2 = \vec{\varepsilon}_2\cos(\alpha y) + \vec{\varepsilon}_4\sin(\alpha y),$$
$$\vec{e}_4 = \vec{\varepsilon}_2\sin(\alpha y) - \vec{\varepsilon}_4\cos(\alpha y),$$
$$d\vec{r} = \vec{\varepsilon}_1 dt + \omega^2(\vec{\varepsilon}_2\cos(\alpha y) + \vec{\varepsilon}_4\sin(\alpha y)) +$$
$$+ \vec{\varepsilon}_3 dy + \omega^4(\vec{\varepsilon}_2\sin(\alpha y) - \vec{\varepsilon}_4\cos(\alpha y)),$$

where $(\vec{\varepsilon}_1,\vec{\varepsilon}_2,\vec{\varepsilon}_3,\vec{\varepsilon}_4)$ is the constant orthonormal basis, α=const≠0. Note that

$$d(\cos(\alpha y)\omega^2 + \sin(\alpha y)\omega^4) = 0,$$
$$d(\sin(\alpha y)\omega^2 - \cos(\alpha y)\omega^4) = 0.$$

Consequently, one can put

$$\cos(\alpha y)\omega^2 + \sin(\alpha y)\omega^4 = dx,$$
$$\sin(\alpha y)\omega^2 - \cos(\alpha y)\omega^4 = dz.$$

Hence,

$$dr = \vec{\varepsilon}_1 dt + \vec{\varepsilon}_2 dx + \vec{\varepsilon}_3 dy + \vec{\varepsilon}_4 dz.$$

From here we find

$$\vec{r} = t\vec{\varepsilon}_1 + x\vec{\varepsilon}_2 + y\vec{\varepsilon}_3 + z\vec{\varepsilon}_4 + \vec{r}_0, \qquad (17$$

$\vec{r}_0$ – constant vector. Place origin of the fixed coordinate system in the point $M_0(\vec{r}_0)$, recognize the vectors $(\vec{\varepsilon}_1,\vec{\varepsilon}_2,\vec{\varepsilon}_3,\vec{\varepsilon}_4)$ as a basis. From (17) it follows that any point $M\in \boldsymbol{E}_4$ in the given fixed Cartesian coordinate system has the coordinates (t,x,y,z), but the vector field, meeting the theorem conditions, is the field $\vec{e}_4=\sin(\alpha y)\vec{\varepsilon}_2-\cos(\alpha y)\vec{\varepsilon}_4$, where α=const≠0. By doing so it is proved that in $\boldsymbol{E}_4$ there is the only vector field, for which all three principal curvatures of the 2-nd kind are zeroes, the lines of flow are the straight lines, but nonholonomicity vector is the constant vector. The Pfaffian variety orthogonal to the given field, nonholonomic and is determined by the Pfaffian equation

$$\sin(\alpha y)dx - \cos(\alpha y)dz = 0,$$

nonholonomicity vector is the vector $\vec{\rho} = \frac{\alpha}{2}\vec{\varepsilon}_1$.

For the vector field $\vec{e}_4=\sin(\alpha y)\vec{\varepsilon}_2-\cos(\alpha y)\vec{\varepsilon}_4$ we find fundamental invariants, invariant lines and surfaces in the Cartesian fixed coordinate system.

Equidirectional surfaces are three-dimensional planes $y=c$.

Asymptotic lines are lines lying in two-dimensional planes

$$x = a,$$
$$z = b \qquad (18)$$

and

$$y = c,$$
$$z = \text{tg}(\alpha c)x + m. \qquad (19)$$

The principal curvatures of the 1-st kind will be $k_1^{(1)} = 0, k_2^{(2)} = \frac{\alpha}{2}, k_3^{(3)} = -\frac{\alpha}{2}$. The curvature lines of the 1-st kind corresponding to the curvature $k_1^{(1)}$=0, are the straight lines being the lines of plane intersection (18) and (19). The curvature lines of 1-st kind, corresponding to the curvatures $k_2^{(1)} = \frac{\alpha}{2}, k_3^{(1)} = -\frac{\alpha}{2}$, are helical lines, determined by the equations

$$t = c_1,$$
$$x = \frac{1}{\alpha}\sin(\alpha y) + c_2,$$
$$z = -\frac{1}{\alpha}\cos(\alpha y) + c_3 \qquad (20)$$

and

$$t = c_1,$$
$$x = -\frac{1}{\alpha}\sin(\alpha y) + c_2,$$
$$z = \frac{1}{\alpha}\cos(\alpha y) + c_3. \qquad (21)$$

From (20) and (21) we see that curvature lines of the 1-st kind, going through the point $M \in \boldsymbol{E}_4$ and corresponding to the principal curvatures of the 1-st kind not equal to zero belong to one tree-dimensional plane, lie on two circular cylinder of the same radius $\frac{1}{\alpha}$ with common generator and common two-dimensional diametral plane. The curvatures k of all curvature lines of the 1-st kind are the same ($k = \frac{1}{\alpha}$). Torsions of these lines $\kappa = -\frac{1}{\alpha}$, i.e. are also the same in all points.

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APPLICATION OF ONE-DIMENSION STS-DISTRIBUTION FOR MODELLING MAGNITUDES OF STOCK INDEXES

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Modified method STS-GARCH(1,1) has been considered. Modification consisted in rejection of the statement on normal low of lagarithm distribution of time series day increment and in their application for the description of Smoothly Truncated a-Stable (STS)-distribution (smoothly abridged a-stable). The method parameters were found by the technique of maximum likelihood. Statistic investigation of the suggested algorithm accuracy was carried out and decrease of autocorrelation in data structure used for the analysis was shown. The method was used to predict share prices of lag 5.

1. Introduction

Study of properties, calculation of parameters and determination of distribution type of some stochastic process underling the market fluctuations is a main problem of econometrics. Information on distribution is necessary for designing econometric methods (ARCH, GARCH, EGARCH, FIGARCH, FIEGARCH etc., see on the methods in details in [1]), estimation of risk limit value VAR, calculation of probable values of dynamic series in future as well as defining asymptomatic behaviour of distribution function densities. The latter is of particular importance since rare events determining forms and type of their tails correspond to making the most possible profit or suffering the most probable losses.

In most cases logarithms of day increments in financial tool quotations (shares, bonds, swaps, options etc.) do not have normal distribution [2–4]. It is connected with the fact that empirical density distribution function designed on such logarithms has a non-zero excess and asymmetry; there is an oblongness of density function in ε-suburb of mathematical expectation point as well as the so-called «thick tails» (in case when probability of significant changes in prices is higher than that of normal distribution) are observed. All these factors make difficult or impossible to apply the common econometric methods: ARCH(p), GARCH(p,q) and others, which are initially based on the assumption of normal increment and remainder distribution.

Dissatisfaction of financial market participants with the results obtained by normal approximation make the researchers search for new distributions and develop new approaches to empirical financial data processing. Thus, in the works [5–7] to describe time series the Pareto generalized distribution, in [8, 9] the Student generalized t-distribution, in [3] the Laplace distribution, in [10] α-stable distribution has been used. However, at present the idea of combination of all mentioned distributions with normal one is developing increasingly (see, for example, [11]). The idea consists in cutting off the