ORTHOGONAL-BASED HYBRID BLOCK METHOD FOR SOLVING SECOND ORDER INITIAL AND BOUNDARY VALUE PROBLEMS

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ГИБРИДНЫЙ БЛОЧНЫЙ МЕТОД ДЛЯ РЕШЕНИЯ НАЧАЛЬНЫХ И ГРАНИЧНЫХ ЗАДАЧ СО ВТОРОЙ ПРОИЗВОДНОЙ

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Аннотация. *В настоящей работе мы рассматривает метод прямого численного интегрирования начальной и краевой задач второго порядка. С использованием метода коллокаций мы получили новый класс ортогональных базисных полиномов и разработали двухшаговый гибридный блочный метод, который позволяет находить интеграл дифференциального уравнения без его редукции к системе уравнений. В ходе исследований установлен порядок точности, сходимость и область абсолютной устойчивости предлагаемого метода. Приведены результаты численных экспериментов, которые демонстрируют применимость и вычислительную эффективность предложенного метода.*

Introduction. Second order differential equation of the form $y''=f(x,y,y')$ with initial conditions $y'(x_0)=z_0$, $y(x_0)=y_0$ or $y(a)=\alpha$, $y(b)=\gamma$ arise frequently in areas of science, engineering and technology. Early method involves reducing the second order differential equation to a system of first order before solving them with existing method [1-3] were able to solve problems of this type by reducing the problem of second order to first order, however the process is time consuming and rigours to implement. Some of second order differential equations are known to have no analytical solution, and an exact (analytical) solution of boundary value problems (BVPs) is more difficult than a solution of initial value (Cauchy) problems (IVPs). Therefore, researchers are giving increased interest to develop approximate methods for solving such problems. By the type of representation of the results of an approximate solution, the methods can be divided into two groups. The first group of numerical schemes give an approximate solution on the interval [a, b] in the form of a certain function, and the second group introduces the skeleton of an approximate solution on a grid of interval [a, b]. We will follow the common classification of approximate methods: a) reduction methods to the Cauchy problem (shooting method, differential method, the reduction method); b) finite difference method; c) balances method or integro-interpolation method; d) projection-difference methods (finite element method); e) collocation method; f) projection methods (moments method, Galerkin method); g) variation methods (least squares methods, Ritz method). But direct methods which are self-starting and take less computation time are developed in terms of linear multistep methods (LMMs) which are called block methods. In the paper [4] authors used the self-starting scheme to derive a class of one-step hybrid methods for the numerical solution of second order differential equation with power series. In this study, we develop a two-step hybrid block method with orthogonal polynomials as our basis function using collocation technique.

Construction of Orthogonal Polynomial Basis Function. A family of polynomial is said to be orthogonal if the inner product of any two distinct polynomial equals zero. The orthogonal polynomial defined over [-1, 1] with respect to weight function $w(x)=x^2$ as $\varphi_r(x)=\sum_{r=0}^{n} C_r^n x^r$ with the addition property: $\varphi_n(1)=1$, satisfying the requirement $<\varphi_m(x),\varphi_n(x)>=0$, $m>n$, $m=n+1$. From these definitions the following orthogonal polynomial are generated $\varphi_0(x)=1$, $\varphi_1(x)=x$, $\varphi_2(x)=\frac{1}{2}\left(5x^2-3\right)$, $\varphi_3(x)=\frac{1}{2}\left(7x^3-5x\right)$, $\varphi_4(x)=\frac{1}{8}\left(63x^4-70x^2+15\right)$ and $\varphi_5(x)=\frac{1}{8}\left(99x^5-126x^3+35x\right)$. These polynomials are employed as basis function for the derived scheme, where convergence, consistence, and order and error constants were determined.

Development of the Method. We seek to derive numerical scheme using LMM

$$\sum_{i=k-2}^{k}\alpha_i y_{n+1}=h^2\sum_{i=0}^{k}\beta_i f_{n+1}+h^2\beta_v f_v \qquad (1)$$

where k is the number of blocks, h is step of the method, α_i, β_v, v={0, 1, 3/2, 2} are the real unknown parameters to be determined. We express the approximation of the analytical solution of the problem with a polynomial of the form

$$y(x)=\sum_{i=0}^{r+s-1}\alpha_i\varphi_i(x) \qquad (2)$$

where $\varphi_i(x)$ are the derived orthogonal polynomials, r is the number of collocation points, s is the number of interpolation points. We interpolate at interval [0, 1] and collocate at points v={0, 1, 3/2, 2}. From the interpolation and collocation points, we obtained a system of 6 equations each of order 5. From equations (1) and (2) we obtain the continuous scheme:

$$E=\alpha_0 y_n+\alpha_1 y_{n+1}+\alpha_2 f_n+\alpha_3 f_{n+1}+\alpha_4 f_{n+3/2}+\alpha_5 f_{n+2}\,. \qquad (3)$$

From the scheme (3), taking $t=x-x_n$ the value of the α_i were obtained using matrix inversion algorithms. The explicit form of the parameters α_i will be given in the oral presentation. Substituting the value of α_i into the scheme (3) and evaluating at x=3/2 and 2 yields the following implicit discrete scheme:

$$y_{n+2}=-y_n+2y_{n+1}+\frac{1}{12}h^2 f_n+\frac{5}{6}h^2 f_{n+1}+\frac{1}{12}h^2 f_{n+2}, \qquad (5)$$

$$y_{n+\frac{3}{2}}=-\frac{1}{2}y_n+\frac{3}{2}y_{n+1}+\frac{1}{24}h^2 f_n+\frac{13}{32}h^2 f_{n+1}-\frac{5}{48}h^2 f_{n+\frac{3}{2}}+\frac{1}{32}h^2 f_{n+2}\,.$$

Differentiating the continuous scheme with respect to x and evaluating at x_n, x_{n+1}, $x_{n+3/2}$, x_{n+2}yields the following discrete scheme:

$$z_n=-\frac{y_n}{h}+\frac{y_{n+1}}{h}-\frac{89}{360}hf_n-\frac{189}{360}hf_{n+1}-\frac{33}{360}hf_{n+2}+\frac{128}{360}hf_{n+\frac{3}{2}}, \qquad (6)$$

$$z_{n+1} = -\frac{y_n}{h} + \frac{y_{n+1}}{h} + \frac{31}{360}hf_n + \frac{234}{360}hf_{n+1} + \frac{27}{360}hf_{n+2} - \frac{112}{360}hf_{n+\frac{3}{2}},$$

$$z_{n+\frac{3}{2}} = -\frac{y_n}{h} + \frac{y_{n+1}}{h} + \frac{233}{2880}hf_n + \frac{2562}{2880}hf_{n+1} + \frac{141}{2880}hf_{n+2} - \frac{56}{2880}hf_{n+\frac{3}{2}},$$

$$z_{n+2} = -\frac{y_n}{h} + \frac{y_{n+1}}{h} + \frac{31}{360}hf_n + \frac{294}{360}hf_{n+1} + \frac{87}{360}hf_{n+2} + \frac{128}{360}hf_{n+\frac{3}{2}}.$$

Equations (5)-(6) yield our desired block method that is self-starting method.

Order and Error Constant of Proposed Method. We define local truncation error associated with a second order differential equation by the difference operator $L[y(x):h] = \sum_{i=0}^{k}[\alpha_i y(x_n + ih) - h^2 \beta_j f(x_n + ih)]$, here $y(x)$ is an arbitrary function, continuously differentiable on the interval $[a,b]$. Expanding the above expression $L[y(x):h]$ in the Taylors series about the point x, we obtain: $L[y(x):h] = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \ldots + C_{p+2} h^{p+2} y^{p+2}(x)$, where $C_0 = \sum_{i=0}^{k}\alpha_i$, $C_1 = \sum_{i=0}^{k} i\alpha_i$, $C_2 = \frac{1}{2!}\sum_{i=0}^{k} i\alpha_i - \beta_i$ and $C_q = \frac{1}{q!}\sum_{i=0}^{k} i^q \alpha_i - q(q-1)(q-2)i^{q-2}\beta_i$. According to Lambert [2] the order is p if $C_0 = C_1 = C_2 = \ldots = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$, and $C_{p+2} h^{p+2} y^{p+2}(x_n)$ is called the local truncation error at the point x_n. The equations (5), (6) are of order p=4 with the error constant C_{p+2}=[–1/240, –21/10240]^T and C_{p+2}=[–9/4, –61/8, –171/16, –7/4]^T respectively.

Numerical Examples. We consider four numerical examples: Van Der Pol Oscillator Problem [5], IVP of Bratu-type [6], Troshe's Problem [7] and nonlinear system of BVP [8] to test the efficiency of the derived orthogonal-based hybrid block method.

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