CONSTRUCTING A RISKY OPTIMAL MEAN/VALUE-AT-RISK PORTFOLIO

O.A. Belsner, O.L. Kritski
Tomsk Polytechnic University, Russia, Tomsk, Lenin Ave, 30, 634050
E-mail: olegkol@tpu.ru

ФОРМИРОВАНИЕ ОПТИМАЛЬНОГО ПОРТФЕЛЯ ПО СООТНОШЕНИЮ ДОХОДНОСТЬ/ПРЕДЕЛЬНАЯ ВЕЛИЧИНА РИСКА

О.А. Бельснер, О.Л. Крицкий
Национальный исследовательский Томский политехнический университет,
Россия, г. Томск, пр. Ленина, 30, 634050
E-mail: olegkol@tpu.ru

Аннотация. *Представлена модель формирования портфеля с учетом предельной величины риска, что позволило уменьшить начальные инвестиционные вложения, ослабила влияние резких падений фондового рынка на стоимость портфеля, увеличила реализованную доходность инвестиций при сопоставимом по сравнению с классической методологией Марковица уровне риска. Использование метода Бенати – Рицци удобно для создания широкого спектра инвестиционных портфелей для массового неквалифицированного инвестора с различным профилем неприятия риска*

Introduction. The classical Markowitz model used for construction and managing the investment portfolio has been widely used for more than 50 years now. Nevertheless, along with obvious pluses of this approach there are also some disadvantages. Firstly, calculation of a portfolio shares does not take into account possible changes in the asset parameters and depends only on its current values. Therefore, any boost or fall of the asset price in the future will lead to the leap of a portfolio shares when reconstructing it. Secondly, it is impossible to construct an investment portfolio during the process of structural changes at the stock market when long period of price fall turns into steady growth. In this case historical returns of the prices are negative, so a few number of increasing quotes we observe at this period does not allow us to construct portfolio with positive shares. To overcome the drawback of working with a portfolio return volatilities we propose to use $VaR_\alpha(X_t)$ [1,2]:

$$VaR_\alpha(X_t) = \inf\{x \in \Re, P(X_t < x) \geq \alpha\}, \tag{1}$$

where X_t is a random variable describing future investment yield, α is given as threshold value and $F(x) = P(X_t < x)$ is distribution function X_t.

If $F(x)$ is a continuous strictly monotonic rising function then $F(x)$ has an inverse function and (1) will turn to:

$$VaR_\alpha(X_t) = F^{-1}(\alpha).$$

Despite its wide use as a risk measure, *VaR* is still not that popular in construction of the “return/risk” investment portfolio. Using *VaR* in portfolio construction is quite complicated task due to: 1) its stochastic nature dependent on the analyzed data distribution function; 2) its incoherency in arbitrary modification when total portfolio price *VaR* is greater than the sum of each portfolio asset *VaR*. Moreover, the function optimized during the portfolio shares calculation is not concave, so we do not reach a unique stable solution. This is the reason why some researchers choose other risk measures for portfolio construction, e.g., conditional value-at-risk $CVaR_\alpha$ or $CvaR$ in short. This one is coherent for any probability distribution law:

$$CVaR_\alpha(X_t) = E\{X_t, X_t < VaR_\alpha(X_t)\}.$$

Nevertheless, despite all the apparent disadvantages of the VaR methodology in the context of an optimal risk portfolio task, in the present paper we will use it. In the first place our decision can be explained by the fact that optimization tasks with quintile risk measures are quite complicated. This happens because even if we just simply replace portfolio volatility with VaR, the number of arithmetical operations in the Markowitz algorithm will increase exponentially. Furthermore, under the condition of asset returns normal distribution (elliptical in general case), VaR is coherent risk measure [1], and the optimization task with VaR becomes the classical Markowitz model [3].

Research methods. Let us choose K risky assets at the stock market. Suppose x_i is a random variable that describes portfolio return at the moment i, $1 \le i \le T$, where T is the moment of the portfolio construction, $F(x)$ is a distribution function of x_i. Let R_j be a random variable characterizing the relative asset return j, $1 \le j \le K$, λ_j is its share at the constructed portfolio, r_{ij} is observed return R_j at the moment i, $1 \le i \le$ and $r_{min} = \min\limits_{i,j} \{r_{ij}\}$ is a minimal return level for each assets of the portfolio. Let α be the quintile that fixes VaR according to (1). Finally, r_{VaR} is relative portfolio return set by its manager.

Let us construct optimal portfolio with restrictions in VaR and fixed α. The observed portfolio return in this case will be as follows:

$$x_i = \sum_{j=1}^{K} \lambda_j r_{ij} \; .$$

We will not make any suggestions about density function of relative asset return. To estimate VaR let's use the historical modeling method, so that we could avoid the problem of best distribution function of R_j.

Let us formulate the optimal portfolio with VaR task in the following way (using the Benati-Rizzi method) [4]:

$$\max_{\lambda, x, y} \sum_{i=1}^{T} p_i x_i \; ; \qquad (2)$$

$$x_i = \sum_{j=1}^{K} \lambda_j r_{ij} \, , \; 1 \le i \le T; \qquad (3)$$

$$x_i \ge r_{\min} + (r_{VaR} - r_{\min}) y_i \, , \; 1 \le i \le T; \qquad (4)$$

$$\sum_{i=1}^{T} p_i (1 - y_i) \le \alpha \; ; \qquad (5)$$

$$y_i \in \{0,1\} \, , \; 1 \le i \le T,$$

$$\sum_{j=1}^{K} \lambda_j = 1 \, , \lambda_j \ge 0 \, , \qquad (6)$$

where p_i is the probability of x_i in the set of empirical observations and $1 \le i \le T$.

Notice that y_i in (6) are binary with only zero or unit value. This is important for the correct estimation of a portfolio risk value in (5): each time when x_i is less than r_{VaR}, we suppose y_i to be zero. Therefore, in (4) we summarize only probabilities p_i, for which the observed return x_i is less than VaR. If the sum in (5) is greater than α, our portfolio turns into unexecutable one.

Solution of (2)-(6) is complicated due to exponential calculation difficulties of the applied integer linear programming because for the integer variables y_i, $1 \le i \le T$, there are 2^T possible combinations. That is why in this paper we it is reasonable to use package program IBM ILOG CPLEX Optimization Studio 12.8.

Results. Let us construct the portfolio P1 that consists of the Russian blue chips included in MICEX-10 index. Let us use the Benati-Rizzi method (2)-(6). As the initial data we used quotation of the following companies: RAO Aehroflot, RAO Aviakompaniya ALROSA, RAO Bank VTB, RAO Gazprom, RAO GMK Norilsk Nickel, RAO Lukoil, RAO Magnit, RAO MosBirzha, RAO NK Rosneft, RAO Sberbank. The data ranges through January 3 2017 to December 29 2017 for 252 observations. Suppose the VaR value is 0.95 and then find the solution of (2)-(6) with the IBM CPLEX. So, we have shares as follows: RAO Aehroflot has share 0.15, RAO Gazprom has share 0.37, RAO GMK Norilsk Nickel has share 0.16, RAO Lukoil has share 0.11, RAO Magnit has share 0.05 and RAO Sberbank has share 0.16.

The observed portfolio P1 return at the moment of its construction (January, 29 2017) was 95%, that exceeds the MICEX-10 return at the same period (it leveled at -23.04%). Using the same historical data we calculated asset shares of another portfolio P2 constructed by the Markowitz model. According to the calculations, the portfolio P2 consisted of assets of RAO GMK Norilsk Nickel (share 0.26) and RAO Sberbank (share 0.74) only. The observed return of P2 at the moment of its construction was 30% per year.

Conslusion. The proposed model (2)-(6) allows constructing of an investment portfolio taking into account marginal risk level. Usage of this approach demands lower initial investments, weakens influence of critical market drop on portfolio price and increases realized investment return compared to classical Markowitz model. The Benati-Rizzi method is suitable for construction of wide range of investment portfolios managed by unskilled investors with different risk aversion profiles.

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REFERENCES

1. Artzner, P., Delbaen, F., Eber, J.-M., Heath, D. (1998) Coherent measures of risk. *Mathematical Finance*, vol. 9, no. 3, pp. 203–228.
2. Kritski, O.L., Ulyanova, M.K. (2007) Assessing of multivariate financial risks of stock share portfolio. *Applied econometrics*, vol. 2, no. 4, pp. 3-18. (In Russian)
3. McNeil, A. J., Frey, R., Embrechts, P. (2015) Quantitative Risk Management. Concepts, Techniques and Tools. Revised Edition: Princeton University Press, USA, 699 p.
4. Benati, S., Rizzi, R. (2007) A mixed integer linear programming formulation of the optimal mean/Value-at-Risk portfolio problem. *European Journal of Operational Research*, vol. 176, iss. 1, pp. 423–434.