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ON DISTRIBUTION OF MULTI-DIMENSIONAL PLANES IN THE EUCLIDIAN SPACE

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Reflections of two-dimensional squares of m-planes and normal (n–m)-planes of distribution in, defined by two corresponding functions of two arguments meeting the Cauchy-Riemann conditions have been studied.

Introduction

As it is known [1], distribution on differentiated variety M_p represents one of essential sections of differential-geometrical structures. One of the main problems of linear *m-dimensional* subspace *(m-planes)* distribution L_m in *n-dimensional* homogeneous space is the problem of invariant equipment. In *n-dimensional* Euclidian space E_n this problem becomes trivial as with each *m-plane* L_m associates equipping (normal) $(n-m)$-plane $P_{n-m}\perp L_m$. Therefore there is a problem of full attraction of geometrical properties of pair fields of corresponding linear subspaces L_m and P_{n-m} in E_n.

The given work is devoted to studying of $\Delta^1_{n-m}:M\to L_m$ distribution *of m-planes* L_m in E_n ($m>2$, $n-m>2$), where $M\in E_n$. Two-dimensional planes $L^1_2\subset L_m$ and $P^1_2\subset L_m$, passing through the point A are compared to each point $M\in E_n$. A special attention is paid to displays of planes L_2^1 and P_2^1.

The first item is devoted to the analytical device which is applied in all other items at distribution $\Delta^1_{n,m}:M\to L_m$ studying. In item 2 displays $F_t:L^1_2\to P^1_2$ and $\tilde{F}_t:P^1_2\to L^1_2$ are studied at each fixed direction t, which are defined by corresponding two functions of two arguments. In item 3 cases when displays F_t and $\tilde{F}_t$ are analytical, i. e. functions defining them satisfy conditions of Cauchy-Riemann [4. P. 188–189]. In the same item cases of interrelations between numbers m and n are considered when fields of bidimentional planes $L^1_2\subset L_m$ and $P^1_2\subset L_m$ are defined by invariant image at the assumption that displays F_t and $\tilde{F}_t$ are analytical.

All considerations in the given work have local character, and the functions occurring in the work are assumed analytical.

Designations and terminology correspond to accepted in [1–6].

The results stated in items 1–3 for the general distribution $\Delta^1_{n,m}$ in E_n ($m>2$, $n-m>2$) belong to E.T. Ivlev, in the item 3.2 at $n=6$, $n=m+4$ and $m=4$ belong to A.S. Pshenichnikova, at $n\leq 7$ – to V.K. Barysheva.

1. Analytical device

1.1. Distribution $\Delta^1_{n,m}$

The *n-dimensional* Euclidian space E_n is considered. It is attributed to mobile orthonormal reference point $R=\{\overline{A},\overline{e}_i\}$, $(i,j,k,l=\overline{1,n})$ with derivational formulas and structural equations

$$d\overline{A}=\omega^i\overline{e}_i,\qquad d\overline{e}_i=\omega_i^j\overline{e}_j,$$
$$D\omega^i=\omega^j\wedge\omega_j^i,\quad D\omega_i^k=\omega_i^j\wedge\omega_j^k,\tag{1}$$

where 1-forms ω_i^k satisfy correlations:

$$\omega_i^j+\omega_j^i=0,\tag{2}$$

following from orthonormality conditions of reference point R:

$$\langle\overline{e}_i;\overline{e}_j\rangle=\delta_{ij}=\begin{cases}0,\ i\neq j;\\1,\ i=j.\end{cases}\tag{3}$$

Here and in the further the symbol $\langle\overline{x};\overline{y}\rangle$ designates scalar product of vectors $\overline{x},\overline{y}\in E_n$.

In space E_n we shall set distribution

$$\Delta^1_{n,m}:M\to L_m,\tag{4}$$

where M is the current point of space E_n belonging to corresponding *m-plane* L_m.

To distribution (4) we shall attach orthonormal reference point $R=\{\overline{A},\overline{e}_i\}$ so that

$$M=A,\quad L_m=(\overline{A},\overline{e}_1,\overline{e}_2,\dots,\overline{e}_m).\tag{5}$$

Here the symbol $L_p=(\overline{B},\overline{x}_1,\overline{x}_2,\dots,\overline{x}_p)$ means *p-dimensional* plane in $L_p\subset E_n$, passing through the point $B\in E_n$

in parallel linearly to independent vectors $\overline{x}_1,\overline{x}_2,...,\overline{x}_p$ of Euclidian space E_n. From (5) by virtue of (1) follows, that distribution (4) is defined by differential equations:

$$\omega_{\alpha}^{\hat{\alpha}} = A_{\alpha i}^{\hat{\alpha}}\omega^i, \quad (\alpha,\beta,\gamma=\overline{1,m}; \hat{\alpha},\hat{\beta},\hat{\gamma}=\overline{m+1,n}), \quad (6)$$

where components $A_{\alpha i}^{\hat{\alpha}}$ of internal fundamental geometrical object

$$\Gamma = \{A_{\alpha i}^{\hat{\alpha}}\} \quad (7)$$

of the first order of distribution $\Delta_{n,m}^1$ in G.F. Laplas' sense [2] satisfy to differential equations:

$$\nabla A_{\alpha i}^{\hat{\alpha}} = A_{\alpha ij}^{\hat{\alpha}}\omega^j, \quad A_{\alpha[ij]}^{\hat{\alpha}} = 0. \quad (8)$$

Here and in further the operator ∇ means the following:

$$\nabla H_{a_1a_2}^{a} = dH_{a_1a_2}^{b}\omega_b^a - H_{b_1a_2}^{a}\omega_{a_1}^{b_1} - H_{a_1b_2}^{a}\omega_{a_2}^{b_2},$$

$$\left(\begin{array}{l} a,b,c\in G,\ a_1b_1c_1\in G_1,\ a_2b_2c_2\in G_2,\ G,G_1,G_2\subset N, \\ N \text{ is the set of positive natural numbers}\end{array}\right). \quad (9)$$

From (5) and (1) by virtue of (3) follows, that $(n-m)$-plane is defined in each point $A\in E_n$

$$P_{n-m} = (\overline{A}, \overline{e}_{m+1}, \overline{e}_{m+2},...,\overline{e}_n) \perp L_m. \quad (10)$$

The next distribution is associated with this $(n-m)$-plane

$$\Delta_{n,n-m}^2 : A \to P_{n-m}.$$

From (2) and (6) we obtain

$$\omega_{\hat{\alpha}}^{\alpha} = A_{\hat{\alpha} i}^{\alpha}\omega^i = -\omega_{\alpha}^{\hat{\alpha}} \Rightarrow A_{\alpha i}^{\hat{\alpha}} = -A_{\hat{\alpha} i}^{\alpha}. \quad (11)$$

Let's notice in view of (5) and (10), that in local coordinates x^i of reference point R linear subspaces L_m and P_{n-m} are defined by the equations, accordingly:

$$L_m \Leftrightarrow x^{\hat{\alpha}} = 0; \quad P_{n-m} \Leftrightarrow x^{\alpha} = 0. \quad (12)$$

1.2. Fields of two-dimensional planes $L_2^1 \subset L_m$ and $P_2^1 \subset P_{n-m}$, passing through corresponding points $A\in E_n$

On space E_n as on differentiated variety we shall set fields of geometrical objects

$$g_1 = \{g_{\alpha_1}^{\hat{\alpha}_1}\}, \quad g_2 = \{g_{\alpha_2}^{\hat{\alpha}_2}\}, \quad \text{Rang}[g_{\alpha_1}^{\hat{\alpha}_1}] = \text{Rang}[g_{\alpha_2}^{\hat{\alpha}_2}] = 2,$$

$$\left(\begin{array}{c} \alpha_1,\beta_1,\gamma_1 = 1,2; \hat{\alpha}_1,\hat{\beta}_1,\hat{\gamma}_1 = \overline{3,m}; \\ \alpha_2,\beta_2,\gamma_2 = m+1,m+2; \hat{\alpha}_2,\hat{\beta}_2,\hat{\gamma}_2 = \overline{m+3,n}\end{array}\right), \quad (13)$$

the components of which satisfy the differential equations

$$\nabla g_{\alpha_1}^{\hat{\alpha}_1} + \omega_{\alpha_1}^{\hat{\alpha}_1} = g_{\alpha_1 i}^{\hat{\alpha}_1}\omega^i, \quad \nabla g_{\alpha_2}^{\hat{\alpha}_2} + \omega_{\alpha_2}^{\hat{\alpha}_2} = g_{\alpha_2 i}^{\hat{\alpha}_2}\omega^i. \quad (14)$$

From (5) and (10) in view of (12) – (14) follows, that in each point $A\in E_n$ geometrical objects g_1 and g_2 define orthogonal two-dimensional planes $L_2^1\subset L_m$ and $P_2^1\subset P_{n-m}$, passing through the point A:

$$L_2^1 = (\overline{A}, \overline{\varepsilon}_1, \overline{\varepsilon}_2) \Leftrightarrow x^{\hat{\alpha}_1} = g_{\alpha_1}^{\hat{\alpha}_1}x^{\alpha_1}, \quad x^{\hat{\alpha}} = 0;$$

$$P_2^1 = (\overline{A}, \overline{\varepsilon}_{m+1}, \overline{\varepsilon}_{m+2}) \Leftrightarrow x^{\hat{\alpha}_2} = g_{\alpha_2}^{\hat{\alpha}_2}x^{\alpha_2}, \quad x^{\alpha} = 0. \quad (15)$$

Here corresponding linearly independent vectors $\overline{\varepsilon}_{\alpha_1}$ and $\overline{\varepsilon}_{\alpha_2}$ are defined under the formulas:

$$\overline{\varepsilon}_{\alpha_1} = \overline{e}_{\alpha_1} + g_{\alpha_1}^{\hat{\alpha}_1}\overline{e}_{\hat{\alpha}_1}, \quad \overline{\varepsilon}_{\alpha_2} = \overline{e}_{\alpha_2} + g_{\alpha_2}^{\hat{\alpha}_2}\overline{e}_{\hat{\alpha}_2}. \quad (16)$$

Remark 1.1. From (15) by virtue of (13), (10), (5) and (3) we notice, that in each point $A\in E_n$ perpendicular linear subspaces $L_{m-2}^2\subset L_m (L_2^1 \perp L_{m-2}^2)$ and $P_{n-m-2}^2 \subset P_{n-m} (P_2^1 \perp P_{n-m-2}^2)$, passing through point A, are defined:

$$L_{m-2}^2 = (\overline{A}, \overline{\varepsilon}_3,...,\overline{\varepsilon}_m) \Leftrightarrow x^{\alpha_1} = g_{\hat{\alpha}_2}^{\alpha_1}x^{\hat{\alpha}_1}, \ x^{\hat{\alpha}} = 0;$$

$$P_{n-m-2}^2 = (\overline{A}, \overline{\varepsilon}_{m+3},...,\overline{\varepsilon}_n) \Leftrightarrow x^{\alpha_2} = g_{\hat{\alpha}_2}^{\alpha_2}x^{\hat{\alpha}_2}, \ x^{\alpha} = 0, \quad (17)$$

where

$$\overline{\varepsilon}_{\hat{\alpha}_1} = \overline{e}_{\hat{\alpha}_1} + g_{\hat{\alpha}_1}^{\alpha_1}\overline{e}_{\alpha_1}, \quad \overline{\varepsilon}_{\hat{\alpha}_2} = \overline{e}_{\hat{\alpha}_2} + g_{\hat{\alpha}_2}^{\alpha_2}\overline{e}_{\alpha_2}, \quad (18)$$

at that

$$g_{\hat{\alpha}_1}^{\alpha_1} = -g_{\alpha_1}^{\hat{\alpha}_1}, \quad g_{\hat{\alpha}_2}^{\alpha_2} = -g_{\alpha_2}^{\hat{\alpha}_2}. \quad (19)$$

2. Display of L_2^1 and P_2^1 planes

2.1. Fields of some geometrical objects

By means of components of geometrical objects (7) and (13) to the point $A\in E_n$ we shall set in conformity the following values:

$$g_{\alpha_1 i}^{\hat{\alpha}} = A_{\alpha_1 i}^{\hat{\alpha}} + g_{\alpha_1}^{\hat{\alpha}_1}A_{\hat{\alpha}_1 i}^{\hat{\alpha}}, \quad g_{\hat{\alpha}_2 i}^{\alpha} = A_{\hat{\alpha}_2 i}^{\alpha} + g_{\alpha_2}^{\hat{\alpha}_2}A_{\alpha_2 i}^{\alpha};$$

$$G_{\alpha_1 i}^{\alpha_2} = g_{\alpha_1 i}^{\alpha_2} + g_{\alpha_1 i}^{\hat{\alpha}_2}g_{\hat{\alpha}_2}^{\alpha_2}, \quad G_{\alpha_2 i}^{\alpha_1} = g_{\alpha_2 i}^{\alpha_1} + g_{\alpha_2 i}^{\hat{\alpha}_1}g_{\hat{\alpha}_1}^{\alpha_1}, \quad (20)$$

which by virtue of (11), (8), (9), (13), (14) and (19) satisfy the differential equations:

$$\nabla g_{\alpha_1 i}^{\hat{\alpha}} + A_{\hat{\alpha}_1 i}^{\hat{\alpha}}\omega_{\alpha_1}^{\hat{\alpha}_1} + g_{\alpha_1}^{\hat{\alpha}_1}A_{\beta_1 i}^{\hat{\alpha}}\omega_{\hat{\alpha}_1}^{\beta_1} = g_{\alpha_1 ij}^{\hat{\alpha}}\omega^j,$$

$$\nabla g_{\alpha_2 i}^{\alpha} + A_{\hat{\alpha}_2 i}^{\alpha}\omega_{\alpha_2}^{\hat{\alpha}_2} + g_{\alpha_2}^{\hat{\alpha}_2}A_{\beta_2 i}^{\alpha}\omega_{\hat{\alpha}_2}^{\beta_2} = g_{\alpha_2 ij}^{\alpha}\omega^j,$$

$$\nabla G_{\alpha_1 i}^{\alpha_2} + g_{\hat{\alpha}_1 i}^{\alpha_2}\omega_{\alpha_1}^{\hat{\alpha}_1} + g_{\alpha_1}^{\hat{\alpha}_1}g_{\beta_1 i}^{\alpha_2}\omega_{\hat{\alpha}_1}^{\beta_1} = G_{\alpha_1 ij}^{\alpha_2}\omega^j,$$

$$\nabla G_{\alpha_2 i}^{\alpha_1} + g_{\hat{\alpha}_2 i}^{\alpha_1}\omega_{\alpha_2}^{\hat{\alpha}_2} + g_{\alpha_2}^{\hat{\alpha}_2}g_{\beta_2 i}^{\alpha_1}\omega_{\hat{\alpha}_2}^{\beta_2} = G_{\alpha_2 ij}^{\alpha_1}\omega^j. \quad (21)$$

Here

$$g_{\hat{\alpha}_1 i}^{\alpha_2} = A_{\hat{\alpha}_1 i}^{\alpha_2} + g_{\hat{\alpha}_1}^{\alpha_1}A_{\alpha_1 i}^{\alpha_2}, \quad g_{\hat{\alpha}_2 i}^{\alpha_1} = A_{\hat{\alpha}_2 i}^{\alpha_1} + g_{\hat{\alpha}_2}^{\alpha_2}A_{\alpha_2 i}^{\alpha_1},$$

at that the obvious view values standing at ω^j is insignificant for us.

From (20), (21), (8) and (7) we notice that on variety E_n fields of the following geometrical objects in G.F. Laptev's sense [2] are ascertained:

$$\overset{*}{g}_1 = \{A_{\alpha i}^{\hat{\alpha}}, g_1\}, \quad \overset{*}{g}_2 = \{A_{\hat{\alpha} i}^{\alpha}, g_2\},$$

$$\overset{*}{G}_1 = \Gamma_1 \cup \overset{*}{g}_1, \quad \overset{*}{G}_2 = \Gamma_1 \cup \overset{*}{g}_2. \quad (22)$$

In the following item the displays of planes L_2^1 and P_2^1 will be studied, which are associated with fields of geometrical objects (22).

2.2. Displays $F_t: L_2^1 \to P_2^1$ and $\tilde{F}_t: L_2^1 \to P_2^1$

The curve $k(t)$ passing through the point $A\in E_n$ and defined by the parametrical differential equations, is considered:

$$k(t): \quad \omega^i = t^i\theta, \quad D\theta = \theta\wedge\theta_1, \quad (23)$$

where values t^i at fixed main parameters, i. e. at $\omega^i=0$, satisfy conditions:

$$\delta t^i + \pi^i_j t^j = \overset{0}{\theta}_1 t^i .$$

Here $\pi^j_i=\omega^j_i|_{\omega^j=0}$, δ is a symbol of differentiation by secondary parameters [2], [3], at that $\overset{0}{\theta}_1=\theta_1|_{\omega^j=0}$.

From (1) by virtue of (23) we notice that the line

$$t = (\overline{A}, \overline{t}), \quad \overline{t} = t^i \overline{e}_i \tag{24}$$

with directing vector $\overline{t}$, passing through the point A, is the tangent to the curve $k(t)$ in the point A. Further according to (23) and (24) we shall consider that displacement on the curve $k(t)$ is equivalent to displacement in the direction t.

Point $A\in E_n$ we shall compare with points $X\in L^1_2\subset L_m$ and $Y\in P^1_2\subset P_{n-m}$ that have radius-vectors:

$$\overline{X} = \overline{A} + x^{\alpha_1}\overline{\varepsilon}_{\alpha_1}, \quad \overline{Y} = \overline{A} + x^{\alpha_2}\overline{\varepsilon}_{\alpha_2}. \tag{25}$$

From (23)–(25) in view of (1), (15), (16) and (12) we obtain:

$$\frac{d\overline{X}}{\theta} = (...)^{\alpha}\overline{e}_{\alpha} + t^i(\delta^{\hat{\alpha}}_i + g^{\hat{\alpha}}_{\alpha_1 i}x^{\alpha_1})\overline{e}_{\hat{\alpha}},$$

$$\frac{d\overline{Y}}{\theta} = (...)^{\hat{\alpha}}\overline{e}_{\hat{\alpha}} + t^i(\delta^{\alpha}_i + g^{\alpha}_{\alpha_2 i}y^{\alpha_2})\overline{e}_{\alpha}. \tag{26}$$

Here the symbol (...) designates insignificant values.

From (26) in view of (20), (5), (10), (12), (15)–(19) we notice that in each point $A\in E_n$ the following displays are ascertained:

$$F_t : L^1_2 \to P^1_2 \Leftrightarrow y^{\alpha_2} = (G^{\alpha_2}_{\alpha_1 i}x^{\alpha_1} + \delta^{\alpha_2}_i)t^i,$$

$$\tilde{F}_t : P^1_2 \to L^1_2 \Leftrightarrow x^{\alpha_1} = (G^{\alpha_1}_{\alpha_2 i}y^{\alpha_2} + \delta^{\alpha_1}_i)t^i, \tag{27}$$

corresponding to the direction (24). Geometrically each of the displays (27) is characterized as follows:

$$Y = F_t X = \{T(X)_t \cup L_m \cup P^2_{n-m-2}\} \cap P^1_2,$$

$$X = \tilde{F}_t Y = \{T(Y)_t \cup P_{n-m} \cup L^2_{m-2}\} \cap L^1_2. \tag{28}$$

Here the symbol $T(Z)_t$ designates a tangent to the line $(Z)_t$, described by the point Z along the curve (23) or along the direction (24). We shall notice that in (28) it is supposed, that points $X\in L^1_2\subset L_m$ and $Y\in P^1_2\subset P_{n-m}$ are not focuses of linear subspaces L_m and P_{n-m} along the curve $k(t)$ in sense [5].

3. Analytical displays of $L^1_2\subset L_m$ and $P^1_2\subset P_{n-m}$ planes

3.1. Displays F_{ta} and $\tilde{F}_{ta}$

Let the following display answer each point $A\in E_n$:

$$\psi : L^1_2 \to P^1_2 \Leftrightarrow y^{\alpha_2} = \psi^{\alpha_2}(x^1; x^2), \tag{29}$$

where functions $\psi^{\alpha_2}(x^1;x^2)$ are at least twice continuously differentiated on a plane L^1_2.

Definition 3.1. Display $\psi:L^1_2\to P^1_2$ is called analytical and designated as ψ_a, i.e. $\psi\to\psi_a$, if defining it functions (29) satisfy to Cauchy-Riemann conditions [4. P. 188–189] on the plane L^1_2:

$$\frac{\partial\psi^{m+1}(M)}{\partial x^1} = \frac{\partial\psi^{m+2}(M)}{\partial x^2},$$

$$\frac{\partial\psi^{m+2}(M)}{\partial x^1} = -\frac{\partial\psi^{m+1}(M)}{\partial x^2}, \tag{30}$$

$$M(x^1; x^2) \in L^1_2.$$

From (27) we notice that at each fixed direction $t=(\overline{A},\overline{e}_i)t^i$ each display (27) is defined by two corresponding functions of two arguments. Therefore according to the definition 3.1 from (30) and (27) we obtain, that

$$F_t \to F_{ta} : L^1_2 \to P^1_2 \Leftrightarrow \begin{cases} (G^{m+1}_{1i} - G^{m+2}_{2i})t^i = 0; \\ (G^{m+1}_{2i} + G^{m+2}_{1i})t^i = 0, \end{cases}$$

$$\tilde{F}_t \to \tilde{F}_{ta} : P^1_2 \to L^1_2 \Leftrightarrow \begin{cases} (G^1_{m+1,i} - G^2_{m+2,i})t^i = 0; \\ (G^1_{m+2,i} + G^2_{m+1,i})t^i = 0, \end{cases} \tag{31}$$

(t_i is fixed).

The following theorems take place.

Theorem 3.1. Display $F_t:L^1_2\to P^1_2$ corresponding to a point $A\in E_n$, will be a display of F_{ta} at each fixed $t\in E_n$ then, and only then, when display $\tilde{F}_t:L^1_2\to P^1_2$ will be a display of $\tilde{F}_{ta}$.

The proof of this theorem follows in view of (31), (11), (19) and (20) that

$$G^{m+1}_{1i} - G^{m+2}_{2i} = -G^1_{m+1,i} + G^2_{m+2,i},$$

$$G^{m+1}_{2i} + G^{m+2}_{1i} = -G^1_{m+2,i} - G^2_{m+1,i}. \tag{32}$$

Theorem 3.2. To each pair two-dimensional planes $L^1_2\subset L_m$ and $P^1_2\subset P_{n-m}$ in point $A\in E_n$ in general case, i. e. in case, when a rank of a matrix

$$G = \begin{vmatrix} G^{m+1}_{11} - G^{m+2}_{21} & \dots & G^{m+1}_{1n} - G^{m+2}_{2n} \\ G^{m+2}_{11} + G^{m+1}_{21} & \dots & G^{m+2}_{1n} + G^{m+1}_{2n} \end{vmatrix} \tag{33}$$

in the point A is equal to 2, corresponds $(n-2)$-plane

$$\Gamma_{n-2} = (t \in E_n \,|\, F_t \to F_{ta} \Leftrightarrow \tilde{F}_t \to \tilde{F}_{ta}),$$

passing through point A.

Proof of this theorem follows from (31) in view of the theorem 3.1 and parities (32).

Remark 3.1. In view of (31) and (32) and the theorem 3.1 the $(n-2)$-plane (33) is actually defined in local coordinates of orthonormal reference point R by the equations:

$$\Gamma_{n-2} \Leftrightarrow \begin{cases} (G^{m+1}_{1i} - G^{m+2}_{2i})t^i = 0; \\ (G^{m+1}_{2i} + G^{m+2}_{1i})t^i = 0. \end{cases} \tag{34}$$

3.2. Existence of two-dimensional planes $L^1_2\subset L_m$ and $P^1_2\subset P_{n-m}$ in general case at certain values m and n, when $F_t\to F_{ta}\Leftrightarrow\tilde{F}_t\to\tilde{F}_{ta}$

The following theorems take place.

Theorem 3.3. To each point $A\in E_n$ in general case corresponds at $n<7$ uncountable and at $n=7$ – final number of corresponding pairs of planes $L^1_2\subset L_m$ and $P^1_2\subset P_{n-m}$ such, that

$$F_t \to F_{ta} \qquad (35)$$

at all directions t, belonging to some hyperplane Γ_{n-1}.

Proof. From (15) follows that in each point $A\in E_n$ planes $L_2^1\subset L_m$ and $P_2^1\subset P_{n-m}$ are defined by components of geometrical objects (13), number of which is equal, accordingly:

$$L_2^1: m_1 = 2(m-2); \quad P_2^1: m_2 = 2(n-m-2). \qquad (36)$$

From (34) follows that planes L_2^1 and P_2^1 are talked about in the theorem 3.3, in the only case when the rank of the matrix (33) is equal to 1, i. e., when in view of (20) values $g_{\alpha_1}^{\hat{\alpha}_1}$ and $g_{\alpha_2}^{\hat{\alpha}_2}$ satisfy the algebraic equations:

$$V_b \equiv (G_{1n}^{m+2} + G_{2n}^{m+1})(G_{1b}^{m+1} - G_{2b}^{m+2}) - $$
$$-(G_{1b}^{m+2} + G_{2b}^{m+1})(G_{1n}^{m+1} - G_{2n}^{m+2}) = 0, \qquad (37)$$
$$(b = \overline{1, n-1}).$$

From (36) follows that unknown $g_{\alpha_1}^{\hat{\alpha}_1}$ and $g_{\alpha_2}^{\hat{\alpha}_2}$, which number is equal to

$$m_1+m_2=2(n-4),$$

satisfy $n-1$ the algebraic equations (37) in each point $A\in E_n$. Therefore the statement 1, the one we are talking about in this theorem, is fair.

Let's prove validity of the statement 2.

Let's consider Jacob's matrix of the system (37):

$$\left[\frac{\partial V_b}{\partial g_{\alpha_1}^{\hat{\alpha}_1}}; \ \frac{\partial V_b}{\partial g_{\alpha_2}^{\hat{\alpha}_2}}\right]$$
$$\begin{pmatrix} n = 7; b = \overline{1,6}; \alpha_1, \beta_1 = 1,2; \hat{\alpha}_1, \hat{\beta}_1 = 3,4; \\ \alpha_2, \beta_2 = 5,6; \hat{\alpha}_2, \hat{\beta}_2 = 7 \end{pmatrix}. \qquad (38)$$

Let's calculate the rank of the matrix (38) at $g_{\alpha_1}^{\hat{\alpha}_1}=0$, $g_{\alpha_2}^{\hat{\alpha}_2}=0$. From (38) and (37) by virtue of (19) and (20) we notice that the matrix (38) has a determinant (minor) of the sixth order

$$\det[P_{\tilde{b}b}]. \qquad (39)$$

Here indices possess values:

$$\tilde{b} = \begin{pmatrix}7\\1\end{pmatrix}, \begin{pmatrix}7\\2\end{pmatrix}, \begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}5\\4\end{pmatrix}, \begin{pmatrix}6\\3\end{pmatrix}, \begin{pmatrix}6\\4\end{pmatrix}, \quad b = \overline{1,6},$$

and values $P_{\tilde{b}b}$ are defined under the formulas:

$$P_{1b}^7 = -A_{1b}^7 P_7 - A_{27}^7 Q_b + A_{2b}^7 Q_7 + A_{17}^7 P_b,$$
$$P_{2b}^7 = -A_{2b}^7 P_7 - A_{17}^7 Q_b + A_{1b}^7 Q_7 - A_{27}^7 P_b,$$
$$P_{\hat{\alpha}_1 b}^5 = A_{\hat{\alpha}_1 b}^5 P_7 + A_{\hat{\alpha}_1 7}^6 Q_b - A_{\hat{\alpha}_1 b}^6 Q_7 - A_{\hat{\alpha}_1 7}^5 P_b,$$
$$P_{\hat{\alpha}_1 b}^6 = -A_{\hat{\alpha}_1 b}^6 P_7 + A_{\hat{\alpha}_1 7}^5 Q_b - A_{\hat{\alpha}_1 b}^5 Q_7 + A_{\hat{\alpha}_1 7}^6 P_b,$$
$$P_i = A_{1i}^6 + A_{2i}^5, \quad Q_i = A_{1i}^5 - A_{2i}^6 \qquad (40)$$
$$(b = \overline{1,6}; \quad \hat{\alpha}_1 = 3,4; \quad i = \overline{1,7}).$$

From (40) follows that the determinant (39) in the general case in the point $A\in E_7$ is not equal to zero. It means that the rank of the matrix (38) in the general case is equal to 6. Therefore the system (37) consists of 6 algebraic equations and therefore hy has final number of solutions relatively to $g_{\alpha_1}^{\hat{\alpha}_1}$ and $g_{\alpha_2}^7$.

Theorem 3.3 is proved.

Theorem 3.4. To each plane L_2^1 in a set point $A\in L_m$ at n=6 one plane P_2^1 corresponds so, that (35) takes place at $\forall t\in L_2^1$.

Proof. Three cases are possible at n=6.

1. ***m*=2, *n*=6.**

In this case with respect to (5), (10), (13) and (15) we have

$$L_2^1 = L_2 = (\overline{A}, \overline{e}_1, \overline{e}_2) \Rightarrow g_{\hat{\alpha}_1}^{\alpha_1} = g_{\alpha_1}^{\hat{\alpha}_1} = 0, \quad x^{\hat{\alpha}} = 0,$$
$$P_2^1 \subset P_4 = (\overline{A}, \overline{e}_3, \overline{e}_4, \overline{e}_5, \overline{e}_6),$$
$$P_2^1 \Leftrightarrow x^{\hat{\alpha}_2} = g_{\alpha_2}^{\hat{\alpha}_2} x^{\alpha_2}, \quad x^{\alpha} = 0,$$
$$(\alpha_2 = 3,4; \hat{\alpha}_2 = 5,6).$$

Therefore the parity (35), $\forall t\in L_2^1$ in view of (34) will be carried out only in the case when values $g_{\alpha_2}^{\hat{\alpha}_2}=-g_{\hat{\alpha}_2}^{\alpha_2}$, number of which is equal to 4, satisfy the following system 4 of linear in the general case non-uniform equations:

$$\begin{cases} g_{\hat{\alpha}_2}^3 A_{1\alpha_1}^{\hat{\alpha}_2} - g_{\hat{\alpha}_2}^4 A_{2\alpha_1}^{\hat{\alpha}_2} = A_{2\alpha_1}^4 - A_{1\alpha_1}^3; \\ g_{\hat{\alpha}_2}^3 A_{2\alpha_1}^{\hat{\alpha}_2} + g_{\hat{\alpha}_2}^4 A_{1\alpha_1}^{\hat{\alpha}_2} = -A_{1\alpha_1}^4 - A_{2\alpha_1}^3 \end{cases} \qquad (41)$$
$$(\alpha_1 = 1,2; \hat{\alpha}_2 = 5,6).$$

It is possible to show that the main determinant of the fourth order of the system (41) in point A is not equal to zero identically. Therefore the system (41) in general case in point A allows the only decision regarding $g_{\alpha_2}^{\hat{\alpha}_2}$.

2. ***m*=3, *n*=6.**

In this case indexes accept the following values:

$$\alpha_1, \beta_1 = 1,2; \hat{\alpha}_1, \hat{\beta}_1 = 3; \alpha_2, \beta_2 = 4,5;$$
$$\hat{\alpha}_2, \hat{\beta}_2 = 6; i = \overline{1,6}; \alpha, \beta = 1,2; \hat{\alpha}, \hat{\beta} = 4,5,6,$$

at that

$$L_2^1 \Leftrightarrow x^{\alpha_2} = g_{\hat{\alpha}}^3 x^{\hat{\alpha}}, \quad x^{\hat{\alpha}} = 0,$$
$$P_2^1 \Leftrightarrow x^6 = g_{\alpha_2}^6 x^{\alpha_2}, \quad x^{\alpha} = 0; \quad L_2^1 = (\overline{A}, \overline{e}_3),$$
$$L_3 = (\overline{A}, \overline{e}_1, \overline{e}_2, \overline{e}_3) \Leftrightarrow x^{\hat{\alpha}} = 0,$$
$$P_3 = (\overline{A}, \overline{e}_4, \overline{e}_5, \overline{e}_6) \Leftrightarrow x^{\alpha} = 0. \qquad (42)$$

Let's consider that in a point $A\in E_6$ a plane L_2^1 is set. According to (42) we shall lead such canonization of reference point R, at which

$$L_2^1 = (\overline{A}, \overline{e}_1, \overline{e}_2) \Leftrightarrow x^3 = 0, x^{\hat{\alpha}} = 0 \Leftrightarrow g_{\alpha_1}^3 = -g_3^{\alpha_1} = 0, \quad (43)$$

which by virtue of (14) leads to the differential equations

$$\omega_{\alpha_1}^3 = -\omega_3^{\alpha_1} = A_{\alpha_1 i}^3 \omega^i.$$

It means that the specified fixing of reference point R exists according to [6].

From (34) in view of (43) and (20) we shall conclude that (35), $t=(\overline{A},\overline{e}_3)$, $(x^{\alpha_1}=0, x^{\hat{\alpha}}=0)$ takes place in only case when two values $g_6^{\alpha_2}=-g_{\alpha_2}^6$ satisfy the following two in the general case linear non-uniform equations

$$\begin{cases} g_6^4 A_{13}^6 - g_6^5 A_{23}^6 = A_{23}^5 - A_{13}^4; \\ g_6^4 A_{23}^6 + g_6^5 A_{13}^6 = -A_{23}^4 - A_{13}^5. \end{cases} \qquad (44)$$

The main determinant of the second order of this system, as it is easy to see, is not equal identically to zero in the point A. Therefore the system (44) has the only solution regarding g_6^4 and g_6^5.

3. ***m*=4, *n*=6.**

In this case

$$P_2^1 = P_{6-4} = P_2 = (\overline{A}, \overline{e}_5, \overline{e}_6) \Leftrightarrow g_{\alpha_2}^{\hat{\alpha}_2} = -g_{\hat{\alpha}_2}^{\alpha_2} = 0.$$

Thus the plane P_2^1 is considered to be set, and the plane L_2^1 – defined. Hence, the case 3 is formally the same, as well as the case 1.

Theorem 3.4 is proved.

Theorem 3.5. To each point $A \in E_n$ at $n=m+4$ {at $m=4$} in general case corresponds the number of corresponding planes $L_2^1 \subset L_m$ $\{P_2^1 \subset P_{n-m}\}$ so, that (35) takes place at $\forall t \in L_m$ $\{\forall t \in P_{n-m}\}$.

Proof. From (34) in view of (36), (5), (10), (12) and (15) follows that (35) takes place at $\forall t \in L_m \Leftrightarrow t^{\hat{\alpha}}=0$ $\{\forall t \in P_{n-m} \Leftrightarrow t^{\alpha}=0\}$ in only case when values sizes $g_{\alpha_1}^{\hat{\alpha}_1} = -g_{\hat{\alpha}_1}^{\alpha_1}$ and $g_{\alpha_2}^{\hat{\alpha}_2} = -g_{\hat{\alpha}_2}^{\alpha_2}$ satisfy the following nonlinear algebraic equations:

$$\begin{cases} \varphi_c \equiv G_{1c}^{m+1} - G_{2c}^{m+2} = 0; \\ \psi_c \equiv G_{1c}^{m+2} + G_{2c}^{m+1} = 0, \end{cases} \quad (45)$$

$$(c = \overline{1,m} \Leftarrow \forall t \in L_m;\ c = \overline{m+1,n} \Leftarrow \forall t \in P_{n-m}).$$

Here values $G_{\alpha i}^{\hat{\alpha}}$ are defined under the formulas (20).

From (34) and (36) it is possible to conclude that each system (45) contains identical number $m1+m2=2(n-4)$ of unknown $g_{\hat{\alpha}_1}^{\alpha_1}$ and $g_{\hat{\alpha}_2}^{\alpha_2}$ and equations in the following corresponding cases:

$$\forall t \in L_m (t^{\hat{\alpha}} = 0) \Rightarrow n = m+4 \text{ и } c = \overline{1,m},$$

$$\forall t \in P_{n-m} (t^{\alpha} = 0) \Rightarrow m = 4 \text{ и } c = \overline{m+1,n}.$$

The Jacob's matrix of the system (45) is considered

$$\left[\frac{\partial \varphi_c}{\partial g_{\alpha_1}^{\hat{\alpha}_1}};\ \frac{\partial \varphi_c}{\partial g_{\alpha_2}^{\hat{\alpha}_2}};\ \frac{\partial \psi_c}{\partial g_{\alpha_1}^{\hat{\alpha}_1}};\ \frac{\partial \psi_c}{\partial g_{\alpha_2}^{\hat{\alpha}_2}} \right]. \quad (46)$$

Calculating the rank of the matrix (46), for example, at $g_{\alpha_1}^{\hat{\alpha}_1} = -g_{\hat{\alpha}_1}^{\alpha_1} = 0$, $g_{\alpha_2}^{\hat{\alpha}_2} = -g_{\hat{\alpha}_2}^{\alpha_2} = 0$, we are convinced that the matrix (46) has the following nonzero minors in corresponding cases:

1) $n=m+4$.

$$\det\begin{bmatrix} -A_{2\alpha}^{m+3} & -A_{2\alpha}^{m+4} & A_{1\alpha}^{m+3} & A_{1\alpha}^{m+4} & A_{3\alpha}^{m+2} & \ldots & A_{m\alpha}^{m+2} & A_{3\alpha}^{m+1} & \ldots & A_{m\alpha}^{m+1} \\ A_{1\beta}^{m+3} & A_{1\beta}^{m+4} & A_{2\beta}^{m+3} & A_{2\beta}^{m+4} & A_{3\beta}^{m+1} & \ldots & A_{m\beta}^{m+1} & A_{3\beta}^{m+2} & \ldots & A_{m\beta}^{m+2} \end{bmatrix}$$

$$\begin{pmatrix} \alpha = \overline{1,m} \text{ are the numbers of the first } m \text{ lines;} \\ \beta = \overline{1,m} \text{ are the numbers of the next } m \text{ li nes} \end{pmatrix}.$$

2) $m=4$.

$$\det\begin{bmatrix} -A_{m+2,\hat{\alpha}}^{3} & -A_{m+2,\hat{\alpha}}^{4} & A_{m+1,\hat{\alpha}}^{3} & A_{m+1,\hat{\alpha}}^{4} & A_{m+3,\hat{\alpha}}^{2} & \ldots & A_{n-m,\hat{\alpha}}^{2} & A_{m+3,\hat{\alpha}}^{1} & \ldots & A_{n-m,\hat{\alpha}}^{1} \\ A_{m+1,\hat{\beta}}^{3} & A_{m+1,\hat{\beta}}^{4} & A_{m+2,\hat{\beta}}^{3} & A_{m+2,\hat{\beta}}^{4} & A_{m+3,\hat{\beta}}^{1} & \ldots & A_{n-m,\hat{\beta}}^{1} & A_{m+3,\hat{\beta}}^{2} & \ldots & A_{n-m,\hat{\beta}}^{2} \end{bmatrix}$$

$$\begin{pmatrix} A_{\gamma\hat{\beta}}^{\hat{\gamma}} = -A_{\hat{\gamma}\hat{\beta}}^{\gamma};\ \hat{\alpha} = \overline{m+1,n} \text{ are the numbers of the first } n-m \text{ lines;} \\ \hat{\beta} = \overline{m+1,n} \text{ are the numbers of the next } n-m \text{ lines} \end{pmatrix}.$$

As in the case of 1) {2)} the minor of the order $2m\{2(n-m)\}$ in the general case in point A is not equal to zero identically, then the rank of the matrix (46) in corresponding case is equal to $2m\{2(n-m)\}$. It means that the system (46) in each case consists of algebraically independent equations, and therefore assumes the final number of solutions regarding rather $g_{\alpha_1}^{\hat{\alpha}_1}$ and $g_{\alpha_2}^{\hat{\alpha}_2}$.

Theorem 3.5 is proved.

Remark 3.2. Association of cases $n=m+4$ and $m=4$ of the theorem 3.5 leads to the case $m=4$; $n=8$, i. e. to distribution $\Delta_{8,4}^1$ in E_8.

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