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Decays of bosonic and fermionic modes on a domain wall

A. Yu. Loginov[*]

Tomsk Polytechnic University, 634050 Tomsk, Russia

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The decays of excited bosonic and excited fermionic modes in the external field of the domain wall are studied. The wave functions of the excited fermionic modes are found analytically in the external field approximation. Some properties of the fermionic modes are investigated. The reflection and transmission coefficients are calculated for fermion scattering from the domain wall. Properties of the reflection and transmission coefficients are studied. The decays of the first excited fermionic mode are investigated to the first order in the Yukawa coupling constant. The amplitudes, angular distributions, and widths of these decays are found by analytical and numerical methods. Decays of the excited bosonic mode are also investigated to the first order in the Yukawa and self-interaction coupling constants. The amplitudes, angular distributions, and widths of these decays are obtained analytically and by numerical methods.

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I. INTRODUCTION

Field models with spontaneous symmetry breaking possess topologically nontrivial vacuum structures. The field equations of such models may have topologically stabile soliton solutions known as monopoles, strings, and domain walls. It is known [1] that these objects may play an important role in the evolution of the Universe. In particular, domain walls may be a natural and nonexotic alternative to the most popular candidates of dark energy [2]. Domain walls play a key role in mechanisms of the electroweak baryogenesis [3–7]. Moreover, domain walls of $(4+1)$-dimensional field models are the thick branes in the thick-brane world scenarios based on gravity coupled to scalars in higher-dimensional space-time [8–12].

Topological solitons interact with elementary bosons and fermions of the corresponding field models. In particular, the interaction of scalar mesons, Dirac fermions, and Majorana fermions with domain walls has been the object of various studies (see Ref. [1] and references therein; see also Refs. [13–23]). The characteristic feature of fermion-soliton interactions is the existence of fermionic zero modes [15,24]. The presence of fermionic zero modes in background fields of topological solitons leads to important physical phenomena such as fractional fermionic numbers of fermion-soliton systems [15] and superconducting cosmic strings [25]. The fermionic zero modes also have an important effect on the stability of the electroweak strings [26–28] and on properties of the domain walls [29–31].

The domain wall considered here is the $(3+1)$-dimensional generalization of the classical $(1+1)$-dimensional kink of the ϕ^4 model [13,32]. There are bosonic and fermionic modes in the background field of the domain wall. These modes can be either massless or massive. The massless bosonic and massless fermionic modes of the domain wall correspond to the zero bosonic and zero fermionic modes of the kink, respectively. The massless modes are localized on the domain wall and propagate along the wall's surface at the speed of light. The massive modes of the domain wall correspond to the excited modes of the kink. The massive modes can be either localized or nonlocalized on the domain wall.

The bosonic and fermionic modes of the domain wall correspond to states of mesons and fermions in the second quantization formalism. The mesons and the fermions living on the domain wall can interact with each other. In particular, there is meson-meson, meson-fermion, and fermion-fermion scattering on the domain wall. The fermions can also scatter on the antifermions or annihilate them, producing the final mesons. Finally, massive mesons and massive fermions can decay into particles having lower masses.

In the present paper, we study the decays of the excited fermionic and excited bosonic modes that are localized on the domain wall. Along the way, we obtain the exact analytical expressions of the fermionic wave functions and those of the reflection and transmissions coefficients. The paper is structured as follows: In Sec. II, we briefly describe the Lagrangian, the symmetries, the field equations, and the domain-wall solution of the model. Section III is divided into two subsections. In Sec. III A, the well-known properties of the bosonic modes living on the domain wall are summarized. In Sec. III B, the Dirac equation in the external field of the domain wall is considered. The wave functions of the localized and nonlocalized fermions are found analytically, as well as the expressions of the transmission and reflection coefficients. In Sec. IV, we consider the Lagrangian of the interacting bosonic and fermionic modes in the domain wall's background. In particular, the properties of the Lagrangian under the parity transformation are investigated. Then, in Sec. IV A, we

[*]aloginov@tpu.ru

study the decays of the first excited fermionic mode to the first order in the Yukawa coupling constant. The decays of the excited bosonic mode are investigated in Sec. IV B. Finally, in Sec. V, we summarize the results and compare properties of the domain wall's massless modes to those of the kink's zero modes.

Throughout the paper, the natural units $c = 1$, $\hbar = 1$ are used.

II. LAGRANGIAN AND FIELD EQUATIONS OF THE MODEL

The model we are interested in is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 + i\bar{\psi}\gamma^\mu \partial_\mu \psi - g\phi\bar{\psi}\psi, \qquad (1)$$

where ϕ is the real scalar field, ψ is the Dirac fermion field, and γ^μ are the Dirac matrices that satisfy the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ being the space-time metric. The explicit representation [33] of the Dirac matrices that we adopt is chiral:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$
$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (2)$$

where $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and σ^i are the Pauli spin matrices. The Lagrangian density (1) depends on the three parameters λ, η, and g, where λ is the coupling constant of the scalar quartic self-interaction, $\pm\eta$ are the two classical vacuum values of the real scalar field ϕ, and g is the Yukawa coupling constant that is assumed to be positive.

The Lagrangian density is invariant under the global phase rotations of the fermion field: $\psi \to \exp(\alpha)\psi$. The corresponding Noether current is

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad \partial_\mu j^\mu = 0. \qquad (3)$$

The Lagrangian density (1) is also invariant under the discrete $\mathbb{Z}_2$ transformation:

$$\phi \to -\phi, \qquad \psi \to \gamma^5\psi. \qquad (4)$$

By varying the action $S = \int \mathcal{L} d^4x$ in ϕ and $\bar{\psi}$, we obtain the field equations of the model:

$$\partial_\mu\partial^\mu\phi + \lambda(\phi^2 - \eta^2)\phi + g\bar{\psi}\psi = 0, \qquad (5)$$

$$i\gamma^\mu\partial_\mu\psi - g\phi\psi = 0. \qquad (6)$$

Let us consider the static solutions in the bosonic sector ($\psi = 0$) of model (1). It is clear that Eq. (5) has two trivial vacuum solutions $\phi(t, \mathbf{x}) = \pm\eta$, which are related to each other by $\mathbb{Z}_2$ transformation (4). However, it is well known [33,34] that the $(1 + 1)$-dimensional version of model (1) has two nontrivial static soliton solutions:

$$\phi(x) = \pm\eta \tanh\left(\frac{x}{w}\right), \qquad (7)$$

where $w = \sqrt{2}\lambda^{-1/2}\eta^{-1}$ is the effective width of the soliton. The solution with the upper (lower) sign is called the kink (antikink). The kink and antikink are also related to each other by $\mathbb{Z}_2$ transformation (4). These solutions interpolate between the classical vacua $\pm\eta$ and are absolutely stable.

In $3 + 1$ dimensions, the kink (antikink) solution (7) corresponds to the domain wall (antiwall). The domain wall (antiwall) is extended in two spatial dimensions and has infinite energy. The surface energy density ε of the domain wall (antiwall) is

$$\varepsilon = \frac{4}{3}\frac{\eta^2}{w}. \qquad (8)$$

III. BOSONIC AND FERMIONIC MODES ON THE DOMAIN WALL

It is well known [33,34] that the $(1 + 1)$-dimensional kink possesses the bosonic and fermionic modes. These modes can be localized or nonlocalized on the kink. In the case of the $(3 + 1)$-dimensional domain wall, these modes can propagate over the domain wall's plane. In the second quantization formalism, the modes correspond to mesons and fermions propagating on the domain wall. The mesons and fermions living on the domain wall can interact with each other. In particular, excited bosonic and fermionic modes can decay to other modes. Now we consider the bosonic and fermionic modes on the domain wall.

A. Bosonic modes

There are three types of bosonic modes in the external field of the domain wall (7). Let us denote the wave functions of these modes by χ_0, χ_1, and χ_{k}. The wave functions χ_0, χ_1, and χ_{k} are well known [33,34]; we now summarize the results:

$$\chi_0(t, \mathbf{x}, \mathbf{k}_\parallel) = N_0 \exp\left[-i(\omega_0 t - k_y y - k_z z)\right]\mathrm{sech}^2(\xi), \qquad (9)$$

$$\chi_1(t, \mathbf{x}, \mathbf{k}_\parallel) = N_1 \exp\left[-i(\omega_1 t - k_y y - k_z z)\right] \times \sinh(\xi)\mathrm{sech}^2(\xi), \qquad (10)$$

$$\chi_{\mathrm{k}}(t, \mathbf{x}, \mathbf{k}) = N_{\mathrm{k}} \exp\left[-i(\omega_{\mathrm{k}} t - k_x x - k_y y - k_z z)\right] \times (3\tanh^2(\xi) - 1 - w^2k_x^2 - 3iwk_x\tanh(\xi)), \qquad (11)$$

where $\xi = x/w$ is the dimensionless x-coordinate and N_0, N_1, and N_{k} are the normalization constants of the wave functions. In Eqs. (9)–(11), the energies ω_0, ω_1, and ω_{k} are related to the momenta $\mathbf{k}_\parallel = (k_y, k_z)$ and $\mathbf{k} = (k_x, k_y, k_z)$ by the dispersion relations:

$$\omega_0^2 = k_y^2 + k_z^2,$$
$$\omega_1^2 = \frac{3}{4}m_\chi^2 + k_y^2 + k_z^2,$$
$$\omega_{\mathrm{k}}^2 = m_\chi^2 + k_x^2 + k_y^2 + k_z^2, \tag{12}$$

where $m_\chi = 2/w$ is the meson mass. From Eqs. (9) and (10), it follows that the wave functions χ_0 and χ_1 correspond to scalar mesons propagating on the domain wall. In particular, the wave function χ_0 describes the massless scalar mesons propagating on the domain wall at the speed of light. In what follows, we normalize the wave functions χ_0 and χ_1 so that the number of corresponding scalar mesons per unit area of the wall is equal to unity. In this case, the normalization constants N_0 and N_1 are

$$N_0 = \frac{\sqrt{3}}{2\sqrt{w}}\frac{1}{\sqrt{2\omega_0}}, \qquad N_1 = \frac{\sqrt{3}}{\sqrt{2w}}\frac{1}{\sqrt{2\omega_1}}. \tag{13}$$

In contrast to χ_0 and χ_1, the wave function χ_{k} is not localized on the domain wall. Instead, the wave function χ_{k} corresponds to the scalar mesons propagating over all three-dimensional space. We normalize the wave function χ_{k} so that the number of corresponding scalar mesons per unit volume equals unity as $|x| \to \infty$. Then we have the following expression for the normalization constant:

$$N_{\mathrm{k}} = \frac{1}{\sqrt{(1 + k_x^2 w^2)(4 + k_x^2 w^2)}}\frac{1}{\sqrt{2\omega_{\mathrm{k}}}}. \tag{14}$$

Having the normalization constants N_0, N_1, and N_{k}, we obtain the following orthonormality relations:

$$\int \chi_0^*(t, \mathbf{x}, \mathbf{k}'_\parallel)\chi_0(t, \mathbf{x}, \mathbf{k}_\parallel) d^3x = \frac{(2\pi)^2}{2\omega_0}\delta^{(2)}(\mathbf{k}_\parallel - \mathbf{k}'_\parallel),$$
$$\int \chi_1^*(t, \mathbf{x}, \mathbf{k}'_\parallel)\chi_1(t, \mathbf{x}, \mathbf{k}_\parallel) d^3x = \frac{(2\pi)^2}{2\omega_1}\delta^{(2)}(\mathbf{k}_\parallel - \mathbf{k}'_\parallel),$$
$$\int \chi^*(t, \mathbf{x}, \mathbf{k}')\chi(t, \mathbf{x}, \mathbf{k}) d^3x = \frac{(2\pi)^3}{2\omega_{\mathrm{k}}}\delta^{(3)}(\mathbf{k} - \mathbf{k}'). \tag{15}$$

Needless to say, the wave functions χ_0, χ_1, and χ_{k} are mutually orthogonal.

B. Fermionic modes

Fermionic modes living on the domain wall satisfy the Dirac equation (6). First, let us consider the solutions of the Dirac equation that do not depend on the coordinates y and z. Acting on the Dirac equation by the matrix differential operator $i\gamma^\mu\partial_\mu + g\phi_{\mathrm{w}}(x)$, we obtain the system of second-order differential equations

$$(\partial_x^2 - ig\phi'_{\mathrm{w}}(x)\gamma^1 - g^2\phi_{\mathrm{w}}^2(x) + \epsilon^2)\psi = 0, \tag{16}$$

where $\phi_{\mathrm{w}}(x)$ is the domain-wall solution and ϵ is the energy of a fermionic mode. Note that in Eq. (16), the Hermitian matrix $i\gamma^1$ has the two doubly degenerate eigenvalues: 1 and -1. The corresponding eigenvectors ζ_1^{+1}, ζ_2^{+1}, ζ_1^{-1}, and ζ_2^{-1} are

$$\zeta_1^{\pm 1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \zeta_2^{\pm 1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \pm i \\ 1 \\ 0 \end{pmatrix}. \tag{17}$$

Let us denote by $\psi_{\pm 1}(x) = a_{\pm 1}(x)\zeta_i^{\pm 1}$ the bispinors that are proportional to the eigenvectors of the matrix $i\gamma^1$. Then, substituting the explicit form of the domain-wall solution $\phi_{\mathrm{w}}(x)$ in Eq. (16) and changing x to the dimensionless variable $\xi = x/w$, we obtain the decoupled differential equations for the coefficient functions $a_{\pm 1}$:

$$(\partial_\xi^2 + \nu(\mp 1 + \nu)\mathrm{sech}^2(\xi) + w^2(\epsilon^2 - m_\psi^2))a_{\pm 1} = 0, \tag{18}$$

where $\nu = gw\eta$ is the dimensionless positive combination of the model's parameters and $m_\psi = g\eta$ is the mass of the fermion in the background vacuum field $\phi = \eta$. Differential equation (18) coincides in the form with the one-dimensional Schrödinger equation with the Pöschl-Teller potential $V = -\nu(\nu \mp 1)\mathrm{sech}^2(\xi)$. The eigenfunctions and the eigenvalues of this equation are well known [35], so we conclude that Eq. (18) may have both discrete and continuous eigenvalues ϵ. The former correspond to the fermionic modes that are localized on the domain wall, while the latter correspond to the nonlocalized fermionic modes. Let us consider these two cases separately.

1. Nonlocalized fermionic modes

It is clear that the continuous eigenvalues ϵ satisfy the condition $\epsilon^2 > m_\psi^2$; therefore, we can define the dimensionless positive parameter $\mu = wp = w(\epsilon^2 - m_\psi^2)^{1/2}$. Then, the general solution of Eq. (16) can be written as $\psi = \psi_{-1} + \psi_{+1}$, where

$$\psi_{+1} = \sum_{j=1,2}(c_j^{+1}P_{\nu-1}^{-i\mu}(s) + d_j^{+1}Q_{\nu-1}^{-i\mu}(s))\zeta_j^{+1}, \tag{19}$$

$$\psi_{-1} = \sum_{j=1,2}(c_j^{-1}P_{\nu}^{-i\mu}(s) + d_j^{-1}Q_{\nu}^{-i\mu}(s))\zeta_j^{-1}, \tag{20}$$

and $s = \tanh(\xi)$. In Eqs. (19) and (20), P_a^b is the associated Legendre function of the first kind of type 2, Q_a^b is the associated Legendre function of the second kind of type 2, and $c_j^{\pm 1}$, $d_j^{\pm 1}$ are arbitrary constants. We see that the general solution of the second-order system (16) depends on eight arbitrary constants $c_j^{\pm 1}$, $d_j^{\pm 1}$.

Here, we use the associated Legendre functions P_a^b and Q_a^b determined according to Ref. [36]. A comprehensive list of properties of the associated Legendre functions is provided in Ref. [37]. We only note here that P_a^b and Q_a^b can be expressed in terms of the regularized hypergeometric functions and that P_a^b and Q_a^b are expressed in terms of elementary functions if $a \in \mathbb{Z}$.

The Dirac equation (6) can be written in the Hamiltonian form

$$i\partial_t \psi = \mathcal{H}\psi, \tag{21}$$

where

$$\mathcal{H} = -i\gamma^0 \gamma^i \partial_i + g\gamma^0 \phi_{\mathrm{w}}(x) \tag{22}$$

is the Dirac Hamiltonian. On fermionic field configurations that do not depend on y and z, the Hamiltonian (22) reduces to the form

$$\mathcal{H}_x = -i\gamma^0 \gamma^1 \partial_x + g\gamma^0 \phi_{\mathrm{w}}(x). \tag{23}$$

It can easily be checked that the one-dimensional Hamiltonian $\mathcal{H}_x$ commutes with the x-component of the spin operator $\mathbf{s}$:

$$[\mathcal{H}_x, s_x] = 0, \tag{24}$$

where

$$\mathbf{s} = \frac{1}{2}\mathbf{\Sigma} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{25}$$

for γ-matrix representation (2). Clearly, Eq. (24) arises from the fact that domain-wall solution (7) is invariant under a rotation relative to the x-axis. The Hermitian operator s_x has the two doubly degenerate eigenvalues: $1/2$ and $-1/2$. The corresponding eigenvectors $\zeta_1^{\frac{1}{2}}$, $\zeta_2^{\frac{1}{2}}$, $\zeta_1^{-\frac{1}{2}}$, and $\zeta_2^{-\frac{1}{2}}$ are

$$\zeta_1^{\pm\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \zeta_2^{\pm\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ \pm 1 \\ 1 \end{pmatrix}. \tag{26}$$

From Eq. (24), it follows that x-dependent solutions of the Dirac equation (6) can be classified by the eigenvalues of s_x.

The second-order differential operator of Eq. (16) also commutes with s_x. Hence, the general solution of Eq. (16) can be written as the sum of the particular solutions $\psi_{\frac{1}{2}}$ and $\psi_{-\frac{1}{2}}$ that are the eigenvectors of s_x with the eigenvalues $1/2$ and $-1/2$, respectively. The solutions $\psi_{\frac{1}{2}}$ and $\psi_{-\frac{1}{2}}$ can be written as

$$\begin{aligned} \psi_{\pm\frac{1}{2}} = i(&\alpha_{\pm\frac{1}{2}} P_{\nu-1}^{-i\mu}(s) - \beta_{\pm\frac{1}{2}} P_{\nu}^{-i\mu}(s) \\ &+ \gamma_{\pm\frac{1}{2}} Q_{\nu-1}^{-i\mu}(s) - \delta_{\pm\frac{1}{2}} Q_{\nu}^{-i\mu}(s))\zeta_1^{\pm\frac{1}{2}} \\ &\pm (\alpha_{\pm\frac{1}{2}} P_{\nu-1}^{-i\mu}(s) + \beta_{\pm\frac{1}{2}} P_{\nu}^{-i\mu}(s) \\ &+ \gamma_{\pm\frac{1}{2}} Q_{\nu-1}^{-i\mu}(s) + \delta_{\pm\frac{1}{2}} Q_{\nu}^{-i\mu}(s))\zeta_2^{\pm\frac{1}{2}}. \end{aligned} \tag{27}$$

In Eq. (27), the constants $\alpha_{\pm\frac{1}{2}}$, $\beta_{\pm\frac{1}{2}}$, $\gamma_{\pm\frac{1}{2}}$, and $\delta_{\pm\frac{1}{2}}$ are linear combinations of the constants $c_1^{\pm 1}$, $d_1^{\pm 1}$, $c_2^{\pm 1}$, and $d_2^{\pm 1}$ of Eqs. (19) and (20). Note that not every solution of the system of second-order differential equations (16) is the solution of the Dirac equation (6), which is the system of first-order differential equations. Substituting Eq. (27) into the Dirac equation (6) and using recurrence relations for the associated Legendre functions [37], we find that $\psi_{\frac{1}{2}}$ and $\psi_{-\frac{1}{2}}$ are the solutions of the Dirac equation (6) if the following conditions hold:

$$\alpha_{\frac{1}{2}} = -\frac{\mu + i\nu}{w\epsilon}\beta_{\frac{1}{2}}, \qquad \gamma_{\frac{1}{2}} = -\frac{\mu + i\nu}{w\epsilon}\delta_{\frac{1}{2}} \quad \text{for } \psi_{\frac{1}{2}} \tag{28}$$

and

$$\alpha_{-\frac{1}{2}} = \frac{\mu + i\nu}{w\epsilon}\beta_{-\frac{1}{2}}, \qquad \gamma_{-\frac{1}{2}} = \frac{\mu + i\nu}{w\epsilon}\delta_{-\frac{1}{2}} \quad \text{for } \psi_{-\frac{1}{2}}. \tag{29}$$

We see that the solutions of the Dirac equation (6) for a given s_x are determined by the two parameters: $\beta_{\frac{1}{2}}$, $\gamma_{\frac{1}{2}}$ for $s_x = 1/2$, and $\beta_{-\frac{1}{2}}$, $\gamma_{-\frac{1}{2}}$ for $s_x = -1/2$. For $|\xi| \gg 1$, these solutions are the superpositions of the plane waves propagating along the x-axis in the opposite directions. However, it can be shown that if the condition

$$\delta_{\pm\frac{1}{2}} = -\frac{2\tan(\pi\nu)}{\pi}\beta_{\pm\frac{1}{2}} \tag{30}$$

holds, then the solution corresponds to incoming and reflected waves to the right of the wall and to a transmitted wave to the left. On the other hand, if the other condition

$$\delta_{\pm\frac{1}{2}} = \frac{2i\tanh(\pi\mu)}{\pi}\beta_{\pm\frac{1}{2}} \tag{31}$$

holds, then the solution corresponds to incoming and reflected waves to the left of the wall and to a transmitted wave to the right. Let us denote by $\psi_{\pm\frac{1}{2}}^l$ ($\psi_{\pm\frac{1}{2}}^r$) the solution of the Dirac equation with $s_x = \pm 1/2$ that corresponds to the transmitted wave moving to the left (right) from the domain wall. Then, these solutions can be written as follows:

$$\psi^{l}_{\pm\frac{1}{2}} = a^{l}_{\pm\frac{1}{2}}\zeta_1^{\pm\frac{1}{2}} + b^{l}_{\pm\frac{1}{2}}\zeta_2^{\pm\frac{1}{2}}, \tag{32}$$

where

$$a^{l}_{\pm\frac{1}{2}} = i\mathcal{N}^{l}_{\pm\frac{1}{2}}(-P_{\nu}^{-i\mu}(s) \mp \varkappa P_{\nu-1}^{-i\mu}(s)) + i\mathcal{N}^{l}_{\pm\frac{1}{2}}\frac{2\tan(\pi\nu)}{\pi}(Q_{\nu}^{-i\mu}(s) \pm \varkappa Q_{\nu-1}^{-i\mu}(s)), \tag{33}$$

$$b^{l}_{\pm\frac{1}{2}} = \mathcal{N}^{l}_{\pm\frac{1}{2}}(\pm P_{\nu}^{-i\mu}(s) - \varkappa P_{\nu-1}^{-i\mu}(s)) + \mathcal{N}^{l}_{\pm\frac{1}{2}}\frac{2\tan(\pi\nu)}{\pi}(\mp Q_{\nu}^{-i\mu}(s) + \varkappa Q_{\nu-1}^{-i\mu}(s)); \tag{34}$$

$$\psi^{r}_{\pm\frac{1}{2}} = a^{r}_{\pm\frac{1}{2}}\zeta_1^{\pm\frac{1}{2}} + b^{r}_{\pm\frac{1}{2}}\zeta_2^{\pm\frac{1}{2}}, \tag{35}$$

where

$$a^{r}_{\pm\frac{1}{2}} = i\mathcal{N}^{r}_{\pm\frac{1}{2}}(-P_{\nu}^{-i\mu}(s) \mp \varkappa P_{\nu-1}^{-i\mu}(s)) + \mathcal{N}^{r}_{\pm\frac{1}{2}}\frac{2\tanh(\pi\mu)}{\pi}(Q_{\nu}^{-i\mu}(s) \pm \varkappa Q_{\nu-1}^{-i\mu}(s)), \tag{36}$$

$$b^{r}_{\pm\frac{1}{2}} = \mathcal{N}^{r}_{\pm\frac{1}{2}}(\pm P_{\nu}^{-i\mu}(s) - \varkappa P_{\nu-1}^{-i\mu}(s)) + i\mathcal{N}^{r}_{\pm\frac{1}{2}}\frac{2\tanh(\pi\mu)}{\pi}(\pm Q_{\nu}^{-i\mu}(s) - \varkappa Q_{\nu-1}^{-i\mu}(s)). \tag{37}$$

In Eqs. (32)–(37), the phase factor $\varkappa$ is equal to $(\mu + i\nu)/(w\epsilon) = \exp(i\arctan(\nu/\mu))$. The normalization constants $\mathcal{N}^{l}_{\pm\frac{1}{2}}$ and $\mathcal{N}^{r}_{\pm\frac{1}{2}}$ will be determined later.

The bispinor wave functions (32) and (35) describe the Dirac fermions propagating perpendicular to the domain wall; i.e., along the x-axis. Note that domain wall (7) is invariant under Lorentz boosts along the y- and z-axes. This implies that the wave functions of the fermions moving with the two-dimensional momentum $\mathbf{p}_{\parallel} = (p_y, p_z)$ on the domain wall can be obtained by multiplying wave functions (32) and (35) by the spin-1/2 boost matrix $S(\epsilon, \mathbf{p}_{\parallel})$:

$$\Psi^{l,r}_{\pm\frac{1}{2}}(t, \mathbf{x}, p, \mathbf{p}_{\parallel}) = \exp[-i(\gamma_{\parallel}\epsilon t - p_y y - p_z z)] \times S(\epsilon, \mathbf{p}_{\parallel})\psi^{l,r}_{\pm\frac{1}{2}}(p, x), \tag{38}$$

where $\gamma_{\parallel} = (1 - v_{\parallel}^2)^{-1/2}$, $v_{\parallel} = |\mathbf{p}_{\parallel}|(\epsilon^2 + |\mathbf{p}_{\parallel}|^2)^{-1/2}$, and $\epsilon = (m_{\psi}^2 + p^2)^{1/2}$. The boost matrix $S(\epsilon, \mathbf{p}_{\parallel})$ is Hermitian; for γ-matrix representation (2) it has the form

$$S(\epsilon, \mathbf{p}_{\parallel}) = \exp\left(\frac{\chi_{\parallel}}{2}\boldsymbol{\alpha}\cdot\mathbf{n}_{\parallel}\right) = \begin{pmatrix} A_{-} & 0 \\ 0 & A_{+} \end{pmatrix}, \tag{39}$$

where

$$\chi_{\parallel} = \operatorname{arctanh}(v_{\parallel}), \qquad \mathbf{n}_{\parallel} = \frac{\mathbf{p}_{\parallel}}{|\mathbf{p}_{\parallel}|}, \qquad \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma},$$

$$A_{\pm} = \cosh\left(\frac{\chi_{\parallel}}{2}\right) \pm \boldsymbol{\sigma}\cdot\mathbf{n}_{\parallel}\sinh\left(\frac{\chi_{\parallel}}{2}\right).$$

We normalize bispinor wave functions (38) so that the number of incident fermions per unit volume is equal to unity at large distances from the domain wall. Then we have the following expressions for the normalization constants in Eqs. (32)–(37):

$$\mathcal{N}^{l}_{\pm\frac{1}{2}} = \sqrt{\frac{\pi}{2\gamma_{\parallel}}}|\cos(\pi\nu)|\sqrt{\frac{\mu\sinh(\pi\mu)}{\cosh(2\pi\mu) - \cos(2\pi\nu)}}, \tag{40}$$

$$\mathcal{N}^{r}_{\pm\frac{1}{2}} = \sqrt{\frac{\pi}{2\gamma_{\parallel}}}\cosh(\pi\mu)\sqrt{\frac{\mu\sinh(\pi\mu)}{\cosh(2\pi\mu) - \cos(2\pi\nu)}}. \tag{41}$$

At large distances from the domain wall, wave functions (38) correspond to incident and reflected plane waves on one side of the wall and to a transmitted plane wave on the other side. Using the asymptotic expressions of the associated Legendre functions [37] and calculating the incident, reflected, and transmitted fermionic currents, we obtain the expressions of the reflection and transmission coefficients:

$$\mathcal{R} = \frac{2\sin^2(\pi\nu)}{\cosh(2\pi\mu) - \cos(2\pi\nu)}, \tag{42}$$

$$\mathcal{T} = \frac{2\sinh^2(\pi\mu)}{\cosh(2\pi\mu) - \cos(2\pi\nu)}. \tag{43}$$

It can easily be checked that these coefficients satisfy the unitarity relation

$$\mathcal{R} + \mathcal{T} = 1. \tag{44}$$

Note that expressions (42) and (43) for the reflection and transmission coefficients are similar to those obtained in Ref. [16], and coincide with those obtained in Ref. [21] for the zero gauge coupling constant. Note also that these expressions are valid for all four types of fermionic wave functions $\Psi^{l,r}_{\pm\frac{1}{2}}$, as it should be.

Let us investigate some properties of the coefficients $\mathcal{R}$ and $\mathcal{T}$. For any finite ν and $\mu \to \infty$, we have the following asymptotic expressions for $\mathcal{R}$ and $\mathcal{T}$:

$$\mathcal{R} \sim 4\sin^2(\pi\nu)\exp(-2\pi\mu),$$
$$\mathcal{T} \sim 1 - 4\sin^2(\pi\nu)\exp(-2\pi\mu). \tag{45}$$

From Eq. (45), it follows that for any value of the parameter ν, the coefficients $\mathcal{R}$ and $\mathcal{T}$ tend to zero and unity, respectively, as the parameter μ tends to infinity. We see that almost all the fermionic wave passes through the domain wall when the x-component of the momentum of the incident wave becomes large enough. When the parameter μ tends to zero, the expressions for the coefficients $\mathcal{R}$ and $\mathcal{T}$ take the form

$$\mathcal{R} = 1 - \pi^2 \csc^2(\pi\nu)\mu^2 + O(\mu^4),$$
$$\mathcal{T} = \pi^2 \csc^2(\pi\nu)\mu^2 + O(\mu^4). \tag{46}$$

From Eq. (46), it follows that $|\mathcal{R}| \to 1$ and $\mathcal{T} \to 0$ as μ tends to zero. In this case, almost all the fermionic wave is reflected from the domain wall. Note that Eq. (46) is not valid if the parameter ν is a positive integer. Indeed, from Eqs. (42) and (43), it follows that

$$\mathcal{R} = 0, \qquad \mathcal{T} = 1 \tag{47}$$

for a positive integer ν and arbitrary μ. Thus, the domain wall becomes reflectionless for the fermions as $\nu = 1, 2, 3, \ldots$ Note in this connection that the domain wall is reflectionless for mesons for any value of the model's parameters.

Let us investigate the behavior of coefficients $\mathcal{R}$ and $\mathcal{T}$ as ν tends to some positive integer and μ tends to zero. For this, we use the following representation of the parameters ν and μ:

$$\nu = n + \kappa\sin(\alpha), \qquad \mu = \kappa\cos(\alpha), \tag{48}$$

where n is a positive integer and the parameter κ tends to zero. Then it can be shown that the coefficients $\mathcal{R}$ and $\mathcal{T}$ tend to the limits

$$\lim_{\kappa\to 0}\mathcal{R} = \sin^2(\alpha), \qquad \lim_{\kappa\to 0}\mathcal{T} = \cos^2(\alpha). \tag{49}$$

From Eq. (49), it follows that coefficients $\mathcal{R}$ and $\mathcal{T}$ have nonregular behavior as $\nu \to n$, $\mu \to 0$, because their limiting values depend on the direction angle α.

Now we investigate coefficients $\mathcal{R}$ and $\mathcal{T}$ in the thin-wall limit $w \to 0$. In that connection, we recall the definition of the dimensionless parameters: $\nu = gw\eta$ and $\mu = pw$, where p is the modulus of the x-component of the fermion's momentum. Substituting these definitions in Eqs. (42) and (43) and taking the limit $w \to 0$, we obtain the expressions for $\mathcal{R}$ and $\mathcal{T}$ in the thin-wall limit:

$$\mathcal{R} \underset{w\to 0}{\to} \frac{m_\psi^2}{\epsilon^2}, \qquad \mathcal{T} \underset{w\to 0}{\to} \frac{p^2}{\epsilon^2}. \tag{50}$$

Note that Eq. (50) for the reflection and transmission coefficients coincides with those of Refs. [1,22] obtained within the framework of the thin-wall approximation.

Wave functions (38) describe the fermionic states in the domain-wall background. Now we consider the antifermionic states. In this connection, we should note that the Lagrangian (1) is invariant under the charge-conjugation transformation

$$\Psi \to \Psi^c = U_C\bar{\Psi}, \tag{51}$$

where the charge- conjugation matrix U_C can be chosen as

$$U_C = \gamma^0\gamma^2 = \alpha_y. \tag{52}$$

This implies that the antifermionic states in the domain-wall background are described by the wave functions that are charge conjugate to wave functions (38). It can be shown that the charge-conjugate wave functions can be expressed in terms of the original ones with the change of sign of the parameters ϵ, p, $\mathbf{p}_\parallel$, and s_x:

$$\Psi^{l,rc}_{\pm\frac{1}{2}}(t,\mathbf{x},\epsilon,p,\mathbf{p}_\parallel) \equiv U_C\bar{\Psi}^{l,r}_{\pm\frac{1}{2}}(t,\mathbf{x},\epsilon,p,\mathbf{p}_\parallel)$$
$$= \Psi^{l,r}_{\mp\frac{1}{2}}(t,\mathbf{x},-\epsilon,-p,-\mathbf{p}_\parallel). \tag{53}$$

From Eq. (53), it follows that the twice-repeated charge conjugation leads us to the initial wave function.

2. Localized fermionic modes

Research of the fermionic modes localized on the domain wall is similar to that of the nonlocalized fermionic modes. For this reason, we do not repeat the intermediate steps, and go directly to the final results. First, we consider the localized fermionic modes that do not propagate along the domain wall. The energy levels of such modes are quantized and can be written as

$$\epsilon_n^2 = \frac{n(2\nu - n)}{w^2}, \qquad n = 0, 1, \ldots, [\nu], \tag{54}$$

where $[\nu]$ is the integer part of the parameter $\nu = gw\eta$. Note that Eq. (54) coincides with those obtained in Refs. [13,23] for the energy levels of the fermionic modes in the external field of the $(1+1)$-dimensional kink. From Eq. (54), it follows that the fermionic mode having zero energy (the fermionic zero mode) always exists in the external field of the domain wall. We see that at each integer value of ν, a supplementary bound state emerges from the continuum's lower bound $\epsilon = g\eta = m_\psi$ and continues to exist for large values of ν. We also see that the number of the massive localized fermionic modes at rest is $2[\nu]$ (the factor 2 arises because there are two spin states for each massive fermion).

First, we consider the localized fermionic modes with $\epsilon_n > 0$. As well as nonlocalized fermionic modes, localized fermionic modes can be chosen to be the eigenstates of the operator s_x. Let us denote the wave functions of the localized fermionic modes with $s_x = \pm 1/2$ by $\psi_{\pm\frac{1}{2},n}$. Then we have the following expressions for these wave functions:

$$\psi_{\pm\frac{1}{2},n} = a_{\pm\frac{1}{2},n}\zeta_1^{\pm\frac{1}{2}} + b_{\pm\frac{1}{2},n}\zeta_2^{\pm\frac{1}{2}}, \tag{55}$$

where

$$a_{\pm\frac{1}{2},n} = i\mathcal{N}_{\pm\frac{1}{2},n}(-P_\nu^{-\mu_n}(s) \mp \varkappa_n P_{\nu-1}^{-\mu_n}(s)), \tag{56}$$

$$b_{\pm\frac{1}{2},n} = \mathcal{N}_{\pm\frac{1}{2},n}(\pm P_\nu^{-\mu_n}(s) - \varkappa_n P_{\nu-1}^{-\mu_n}(s)), \tag{57}$$

and $s = \tanh(\xi)$. In Eqs. (56) and (57), the factor $\varkappa_n$ and the parameter μ_n are

$$\varkappa_n = in^{\frac{1}{2}}(2\nu - n)^{-\frac{1}{2}},$$
$$\mu_n = \nu - n, \qquad n = 1, \ldots, [\nu]. \tag{58}$$

By analogy with Eq. (38), we can determine the wave functions of the localized fermionic modes moving with the two-dimensional momentum $\mathbf{p}_\parallel = (p_y, p_z)$ along the domain wall:

$$\Psi_{\pm\frac{1}{2},n}(t,\mathbf{x},\mathbf{p}_\parallel) = \exp\left[-i(\gamma_\parallel \epsilon_n t - p_y y - p_z z)\right] \times S(\epsilon_n, \mathbf{p}_\parallel)\psi_{\pm\frac{1}{2},n}(x), \tag{59}$$

where $\gamma_\parallel = (1 - v_\parallel^2)^{-1/2}$ and $v_\parallel = |\mathbf{p}_\parallel|(\epsilon_n^2 + |\mathbf{p}_\parallel|^2)^{-1/2}$. The boost matrix $S(\epsilon_n, \mathbf{p}_\parallel)$ in Eq. (59) is given by Eq. (39). Wave functions (59) describe the x-localized fermionic states in the domain-wall background. The corresponding antifermionic states are described by the wave functions that are charge conjugate to wave functions (59):

$$\Psi^c_{\pm\frac{1}{2},n}(t,\mathbf{x},\epsilon_n,\mathbf{p}_\parallel) \equiv U_C \bar{\Psi}_{\pm\frac{1}{2},n}(t,\mathbf{x},\epsilon_n,\mathbf{p}_\parallel) = \Psi_{\mp\frac{1}{2},n}(t,\mathbf{x},-\epsilon_n,-\mathbf{p}_\parallel). \tag{60}$$

Now we consider the fermionic modes that correspond to $n = 0$ in Eq. (54). In $(1+1)$ dimensions, such a mode corresponds to a particle having zero energy. In $(3+1)$ dimensions, these modes correspond to massless particles moving along the domain wall at the speed of light. It can be shown [33] that the wave function of the massless fermion moving with the two-dimensional momentum $\mathbf{p}_\parallel = (p_y, p_z)$ along the domain wall can be written as

$$\Psi_0(t,\mathbf{x},|\mathbf{p}_\parallel|,\mathbf{p}_\parallel) = \mathcal{N}_0 \frac{\exp\left[-i(|\mathbf{p}_\parallel| t - p_y y - p_z z)\right]}{\sqrt{2|\mathbf{p}_\parallel|}[\cosh(x/w)]^\nu} \times \begin{pmatrix} (|\mathbf{p}_\parallel| - p_z)^{1/2} \\ -ip_y(|\mathbf{p}_\parallel| - p_z)^{-1/2} \\ p_y(|\mathbf{p}_\parallel| - p_z)^{-1/2} \\ i(|\mathbf{p}_\parallel| - p_z)^{1/2} \end{pmatrix}. \tag{61}$$

The wave functions of massless antifermions are charge conjugate to those of massless fermions:

$$\Psi^c_0(t,\mathbf{x},|\mathbf{p}_\parallel|,\mathbf{p}_\parallel) \equiv U_C \bar{\Psi}_0(t,\mathbf{x},|\mathbf{p}_\parallel|,\mathbf{p}_\parallel) = \Psi_0(t,\mathbf{x},-|\mathbf{p}_\parallel|,-\mathbf{p}_\parallel). \tag{62}$$

The wave functions of massless fermions and antifermions satisfy the following algebraic relations:

$$i\gamma^1\Psi_0(t,\mathbf{x},\mathbf{p}_\parallel) = -\Psi_0(t,\mathbf{x},\mathbf{p}_\parallel),$$
$$i\gamma^1\Psi^c_0(t,\mathbf{x},\mathbf{p}_\parallel) = -\Psi^c_0(t,\mathbf{x},\mathbf{p}_\parallel). \tag{63}$$

Using Eq. (63), we can easily obtain the following relations:

$$\bar{\Psi}_0(t,\mathbf{x},\mathbf{p}_\parallel)\mathcal{O}\Psi_0(t,\mathbf{x},\mathbf{p}'_\parallel) = 0,$$
$$\bar{\Psi}^c_0(t,\mathbf{x},\mathbf{p}_\parallel)\mathcal{O}\Psi^c_0(t,\mathbf{x},\mathbf{p}'_\parallel) = 0,$$
$$\bar{\Psi}_0(t,\mathbf{x},\mathbf{p}_\parallel)\mathcal{O}\Psi^c_0(t,\mathbf{x},\mathbf{p}'_\parallel) = 0,$$
$$\bar{\Psi}^c_0(t,\mathbf{x},\mathbf{p}_\parallel)\mathcal{O}\Psi_0(t,\mathbf{x},\mathbf{p}'_\parallel) = 0, \tag{64}$$

where the matrix operator $\mathcal{O}$ may be any of the matrices $\mathbb{I}$, s_x, $\gamma^0 s_y$, $\gamma^0 s_z$, and $\gamma^0\gamma^5$. Note that the momenta $\mathbf{p}_\parallel$ and $\mathbf{p}'_\parallel$ in Eq. (64) can be different; therefore the relations in Eq. (64) are purely algebraic.

Wave functions (61) and (62) also satisfy the relations:

$$\Psi^\dagger_0(t,\mathbf{x},\mathbf{p}_\parallel)s_x\Psi_0(t,\mathbf{x},\mathbf{p}_\parallel) = 0,$$
$$\Psi^{c\dagger}_0(t,\mathbf{x},\mathbf{p}_\parallel)s_x\Psi^c_0(t,\mathbf{x},\mathbf{p}_\parallel) = 0. \tag{65}$$

From Eq. (64) (with $\mathcal{O} = \gamma^0 s_y$, $\gamma^0 s_z$ and $\mathbf{p}_\parallel = \mathbf{p}'_\parallel$) and Eq. (65), it follows that the massless fermions (antifermions) are completely unpolarized in the domain-wall background. In particular, massless fermions (antifermions) have a zero mean value of the helicity operator $\mathbf{s}\cdot\mathbf{n}_\parallel$.

The wave functions of massless fermions (antifermions) are not eigenvectors of the $(3+1)$-dimensional chirality matrix γ^5, and so they have no definite chirality. Thus, the massless fermions living on the $(2+1)$-dimensional domain wall continue to be $(3+1)$-dimensional Dirac fermions. It should be recalled that the free Dirac fermions living in $(3+1)$ dimensions become Weyl fermions in the massless limit. Unlike the massless fermions living on the domain wall, these Weyl fermions possess definite helicities and chiralities.

The generators $\sigma^{\mu\nu} = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)/2$, where μ, ν is equal to 0,2,3 and γ-matrices are defined in Eq. (2), realize a four-dimensional reducible representation of the $(2+1)$-dimensional Lorentz group acting on the domain wall. Therefore, the wave functions of the massless fermions (antifermions) transform according to this reducible representation. Indeed, it can easily be shown that the first two entries of bispinors (61) and (62) transform independently of the last two entries. The two-dimensional irreducible representation of the $(2+1)$-dimensional Lorentz group can be realized by the following 2×2 matrices: $\gamma^0 = \sigma^1$,

$\gamma^1 = -i\sigma^3$, $\gamma^2 = i\sigma^2$. Note that there is no analog of the chiral matrix γ^5 for this two-dimensional irreducible representation. However, for the four-dimensional reducible representation, the matrix $i\gamma^1$ is the analog of the chiral matrix γ^5. Indeed, the matrix $i\gamma^1$ is Hermitian and anticommutates with the matrices γ^0, γ^2, and γ^3. Therefore, Eq. (63) is the analog of the chirality condition for the four-dimensional reducible representation of the $(2+1)$-dimensional Lorentz group.

The wave functions of the massless fermions and of the massless antifermions have the smooth limit as $\mathbf{p}_\parallel \to 0$:

$$\lim_{\mathbf{p}_\parallel\to 0}\Psi_0(t,\mathbf{x},\mathbf{p}_\parallel) = \lim_{\mathbf{p}_\parallel\to 0}\Psi_0^c(t,\mathbf{x},\mathbf{p}_\parallel) \equiv \Psi_0(x,\varphi) = \mathcal{N}_0\left[\cosh\left(\frac{x}{w}\right)\right]^{-\nu}\begin{pmatrix}\sin(\varphi/2)\\ i\cos(\varphi/2)\\ -\cos(\varphi/2)\\ i\sin(\varphi/2)\end{pmatrix}, \tag{66}$$

where φ is the azimuthal angle of $\mathbf{p}_\parallel$ that is counted from the z-axis in the counterclockwise direction. Thus, the massless modes $\Psi_0(t,\mathbf{x},\mathbf{p}_\parallel)$ and $\Psi_0^c(t,\mathbf{x},\mathbf{p}_\parallel)$ become the mode with zero energy (i.e., the zero mode) $\Psi_0(x,\varphi)$ as $\mathbf{p}_\parallel \to 0$. The zero mode $\Psi_0(x,\varphi)$ is invariant under the charge conjugation:

$$\Psi_0^c(x,\varphi) \equiv U_C\bar{\Psi}_0(x,\varphi) = \Psi_0(x,\varphi). \tag{67}$$

The zero mode $\Psi_0(x,\varphi)$ is parametrized by the azimuthal angle φ, so we can define the following two linear combinations:

$$\Psi_0^\pm(x,\varphi) = \frac{1}{\sqrt{2}}[\Psi_0(x,\varphi) \mp i\Psi_0(x,\varphi+\pi)] = \frac{\mathcal{N}_0}{\sqrt{2}}\left[\cosh\left(\frac{x}{w}\right)\right]^{-\nu}\exp\left(\pm i\frac{\varphi}{2}\right)\begin{pmatrix}\mp i\\ i\\ -1\\ \pm 1\end{pmatrix}. \tag{68}$$

The modes $\Psi_0^+(x,\varphi)$ and $\Psi_0^-(x,\varphi)$ are related to each other by the charge conjugation

$$\Psi_0^{\pm c}(x,\varphi) = U_C\bar{\Psi}_0^\pm(x,\varphi) = \Psi_0^\mp(x,\varphi) \tag{69}$$

and are orthogonal and normalized. From Eq. (68), it follows that the modes $\Psi_0^\pm(x,\varphi)$ are proportional to the phase factors $\exp(\pm\varphi/2)$. Hence, the wave functions $\Psi_0^\pm(x,\varphi_1)$ and $\Psi_0^\pm(x,\varphi_2)$, corresponding to two azimuthal angles φ_1 and φ_2, differ in the phase factor $\exp(\pm i(\varphi_2-\varphi_1)/2)$, and so are physically equivalent. Thus, $\Psi_0^+(x,\varphi)$ is the class of physically equivalent states as well as $\Psi_0^-(x,\varphi)$.

Now, let us consider the normalization constants $\mathcal{N}_{\pm\frac{1}{2},n}$ and $\mathcal{N}_0$ in Eqs. (56), (57), (61), and (62). Wave functions (59) and (61) describe the x-localized fermions moving along the domain wall. Therefore, we normalize these wave functions so that the number of corresponding fermions per unit area of the wall is equal to unity. As a result, we can obtain expressions for the normalization constants. These expressions become rather complicated as n increases, so we only give the expressions for $\mathcal{N}_0$, $\mathcal{N}_{\pm\frac{1}{2},1}$, and $\mathcal{N}_{\pm\frac{1}{2},2}$:

$$\mathcal{N}_0 = \frac{1}{\sqrt{w}}\left[2\mathrm{B}\left(\frac{1}{2},\nu\right)\right]^{-\frac{1}{2}}, \tag{70}$$

$$\mathcal{N}_{\pm\frac{1}{2},1} = \frac{2^{\nu-3}\Gamma(\nu)}{\sqrt{\gamma_\parallel w}} \times\left[\frac{2\nu(\nu-1)}{2^{2\nu-1}\nu\mathrm{B}(\nu,\nu) - {}_2F_1(1,-\nu;\nu;-1)+1}\right]^{\frac{1}{2}}, \tag{71}$$

$$\mathcal{N}_{\pm\frac{1}{2},2} = \frac{2^{\nu-3}\Gamma(\nu+1)}{\sqrt{\gamma_\parallel w}}\left[\frac{\nu-2}{\nu(2\nu-1)}\right]^{\frac{1}{2}} \times[2^{2\nu-1}\nu\mathrm{B}(\nu,\nu) - {}_2F_1(1,-\nu;\nu;-1)+1]^{-\frac{1}{2}}, \tag{72}$$

where $\Gamma(\alpha)$ is the gamma function, $\mathrm{B}(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the beta function, and ${}_2F_1(\alpha,\beta;\gamma;\delta)$ is the Gauss hypergeometric function [37].

Having the normalization constants $\mathcal{N}_0$, $\mathcal{N}_{\pm\frac{1}{2},n}$, $\mathcal{N}^l_{\pm\frac{1}{2}}$, and $\mathcal{N}^r_{\pm\frac{1}{2}}$, we obtain the following orthonormality relations for the wave functions of the fermionic modes:

$$\begin{aligned}&\int\Psi_0^\dagger(t,\mathbf{x},\mathbf{p}'_\parallel)\Psi_0(t,\mathbf{x},\mathbf{p}_\parallel)d^3x = (2\pi)^2\delta^2(\mathbf{p}_\parallel-\mathbf{p}'_\parallel),\\ &\int\Psi^\dagger_{s'_x,n'}(t,\mathbf{x},\mathbf{p}'_\parallel)\Psi_{s_x,n}(t,\mathbf{x},\mathbf{p}_\parallel)d^3x = (2\pi)^2\delta_{nn'}\delta_{s_xs'_x}\delta^2(\mathbf{p}_\parallel-\mathbf{p}'_\parallel),\\ &\int\Psi^{\dagger d'}_{s'_x}(t,\mathbf{x},p',\mathbf{p}'_\parallel)\Psi^d_{s_x}(t,\mathbf{x},p,\mathbf{p}_\parallel)d^3x = (2\pi)^3\delta_{s'_xs_x}\delta_{d'd}\delta(p-p')\delta^2(\mathbf{p}_\parallel-\mathbf{p}'_\parallel),\end{aligned} \tag{73}$$

where the indices d and d' are l or r. Note that the wave functions Ψ_0, $\Psi_{s_x,n}$, and $\Psi^d_{s_x}$ are mutually orthogonal, because they are the eigenfunctions of a self-adjoint operator (16) and correspond to different eigenvalues of this operator.

3. Implication of Levinson's theorem for fermionic modes

In the process of scattering, the transmitted fermionic wave acquires a phase shift δ with respect to the incident fermionic wave. This phase shift depends on the modulus of the x-component of the fermion's momentum, so we can write $\delta = \delta(p)$. The difference of the phase shifts $\delta(0) - \delta(\infty)$ plays an important role in the theory of scattering [38]. In particular, Levinson's theorem [39] establishes a relation between this difference and the number of bound states for a given scattering channel. When the two-dimensional momentum $\mathbf{p}_{\parallel}$ vanishes, the scattering of fermions from the domain wall is effectively one dimensional. For a one-dimensional case, Levinson's theorem has the form [40]

$$\delta(0) - \delta(\infty) = \pi\left(n_b - \frac{1}{2}\right), \tag{74}$$

where $\delta(0) - \delta(\infty)$ and n_b are the difference of the phase shifts and the number of bound states in a given scattering channel, respectively.

Using the asymptotic expressions of the associated Legendre functions [37], we obtain the following expression for the difference of the phase shifts:

$$\delta(0) - \delta(\infty) = \pi\left([\nu] - \frac{1}{2}\Delta(\nu)\right), \tag{75}$$

where $[\nu]$ is the integer part of the parameter ν, and the function $\Delta(\nu)$ is equal to 1 if ν is any positive integer, and is equal to 0 otherwise. Note that Eq. (74) is valid for all four fermionic wave functions $\psi^{l}_{\frac{1}{2}}$, $\psi^{l}_{-\frac{1}{2}}$, $\psi^{r}_{\frac{1}{2}}$, and $\psi^{r}_{-\frac{1}{2}}$ which describe the one-dimensional fermionic scattering from the domain wall.

We wish to show that Eq. (75) is the consequence of Levinson's theorem (74). It follows from the results of this section that there are $2[\nu] + 1$ bound (i.e. x-localized) fermionic states for a given value of ν: the $[\nu]$ states having $s_x = 1/2$, the $[\nu]$ states having $s_x = -1/2$, and the one completely unpolarized fermionic zero mode. The case where ν is a positive integer should be considered separately. It can be shown that in this case, bispinor components of the wave functions $\psi_{\pm\frac{1}{2},\nu}$ tend to constant values as $x \to \pm\infty$. The corresponding "half-bound" threshold states with $\epsilon = m_{\psi}$ contribute with a weight of $1/2$ [41,42] to the number of bound states in Levinson's theorem. The fermionic zero mode is completely unpolarized, so the zero-energy fermion has $s_x = 1/2$ (or $-1/2$) with probability 1/2. Therefore, the fermionic zero mode also contributes with a weight of $1/2$ in the count of bound states having $s_x = 1/2$ (or $-1/2$). Note that a similar situation also holds for the $(1+1)$-dimensional cases [41,42], where the fermionic zero mode in the kink's background also counts as $1/2$ in Levinson's theorem. Thus, the effective number of bound fermionic states having $s_x = 1/2$ (or $-1/2$) is

$$n_b = \frac{1}{2} + [\nu] - \frac{1}{2}\Delta(\nu), \tag{76}$$

where the first term is the contribution of the zero mode, and the last term takes into account a weight of $1/2$ of the "half-bound" threshold state. Substituting Eq. (76) into the right-hand side of Eq. (74), we obtain Eq. (75) for the difference $\delta(0) - \delta(\infty)$. Thus, Eq. (75) is the consequence of Levinson's theorem as it should be. Note that Eq. (75), with the extra overall minus sign on its right-hand side, is valid for the scattering of antifermions from the domain wall. This is the consequence of the C-invariance of Lagrangian (1).

IV. DECAYS OF FERMIONIC AND BOSONIC MODES IN THE EXTERNAL FIELD OF THE DOMAIN WALL

Now, let us consider decays of the excited bosonic mode and of the first excited fermionic mode in the external field of the domain wall. From Eq. (1) and from the representation $\phi(t,\mathbf{x}) = \phi_{\mathrm{w}}(x) + \chi(t,\mathbf{x})$ of the scalar field, it follows that the Lagrangian of the interacting bosonic and fermionic modes has the form

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\chi\partial^{\mu}\chi - \frac{\lambda}{2}(3\phi_{\mathrm{w}}^{2} - \eta^{2})\chi^{2} - \lambda\phi_{\mathrm{w}}(x)\chi^{3} - \frac{\lambda}{4}\chi^{4} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - g\chi\bar{\psi}\psi. \tag{77}$$

We consider the decay of bosonic and fermionic modes to the first order in the coupling constants λ and g, while the interaction of the bosonic and fermionic modes with the domain wall's background is taken into account exactly. We use the wave functions of Sec. III as the coefficient wave functions in the expansions of the second quantized operators $\hat{\chi}(t,\mathbf{x})$ and $\hat{\psi}(t,\mathbf{x})$. Then it can be shown that the second quantized operators $\hat{\chi}(t,\mathbf{x})$ and $\hat{\psi}(t,\mathbf{x})$ obey the canonical commutation (anticommutation) relations:

$$[\hat{\chi}(t,\mathbf{x}), \partial_t\hat{\chi}(t,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{78}$$

$$\{\hat{\psi}_i(t,\mathbf{x}), \hat{\psi}_j^{\dagger}(t,\mathbf{y})\} = \delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{79}$$

Thus, the wave functions of Sec. III are properly normalized and can be used for the calculation of decay amplitudes.

The Lagrangian (77) is invariant under the discrete transformation

$$\mathbf{x} \to -\mathbf{x}, \qquad \chi(\mathbf{x}) \to -\chi(-\mathbf{x}), \qquad \psi(\mathbf{x}) \to i\gamma^{5}\gamma^{0}\psi(-\mathbf{x}). \tag{80}$$

This transformation is the generalization of the usual parity transformation, which takes into account the x-antisymmetry of the domain wall. Later on, we shall need the properties of the x-localized massive modes under transformation (80), which we denote symbolically by P:

$$\begin{aligned} P\Psi_{s_x,n}(t,\mathbf{x},\mathbf{p}_{\parallel}) &\equiv i\gamma^5\gamma^0\Psi_{s_x,n}(t,-\mathbf{x},\mathbf{p}_{\parallel}) \\ &= (-1)^{s_x-\frac{1}{2}+n}\Psi_{s_x,n}(t,\mathbf{x},-\mathbf{p}_{\parallel}), \\ P\Psi^c_{s_x,n}(t,\mathbf{x},\mathbf{p}_{\parallel}) &= (-1)^{s_x+\frac{1}{2}+n}\Psi^c_{s_x,n}(t,\mathbf{x},-\mathbf{p}_{\parallel}), \\ P\chi_0(t,\mathbf{x},\mathbf{k}_{\parallel}) &= -\chi_0(t,\mathbf{x},-\mathbf{k}_{\parallel}), \\ P\chi_1(t,\mathbf{x},\mathbf{k}_{\parallel}) &= \chi_1(t,\mathbf{x},-\mathbf{k}_{\parallel}). \end{aligned} \tag{81}$$

Note that the massive modes at rest are the eigenfunctions of transformation (80).

The Lagrangian (77) depends explicitly on the x-coordinate, so the x-component of the total momentum is not conserved in decays in the external field of the domain wall. In this case, the S-matrix of a decay $i \to f$ is written as [43]

$$\begin{aligned} S_{fi} &= i(2\pi)^3\delta(E_f - E_i)\delta(P_{fy} - P_{iy}) \\ &\quad\times \delta(P_{fz} - P_{iz})T_{fi}, \end{aligned} \tag{82}$$

where T_{fi} is the corresponding decay amplitude. The domain wall is invariant under the rotation about the x-axis; therefore, the x-component J_x of the total angular momentum J is conserved for decays in the external field of the domain wall.

A. Decays of the first excited fermionic mode

The existence of the first excited fermionic mode implies that the parameter ν is greater than 1. In this case, the following decay channels are kinematically allowable:

$$\begin{aligned} f_1 &\to f_0 + \chi_0 \quad \text{for } \nu > 1, \\ f_1 &\to f_0 + \chi_1 \quad \text{for } \nu > 2, \\ f_1 &\to f_0 + \chi_{\mathrm{k}} \quad \text{for } \nu > 5/2, \end{aligned} \tag{83}$$

where f_0 and f_1 are fermions of the massless and first excited modes, respectively, while χ_0, χ_1, and χ_{k} are mesons of the massless, excited, and nonlocalized modes, respectively. From Eq. (77) we obtain the general expression for the first-order S-matrix elements of a decay $f_i \to f_f + \chi_f$:

$$S^{(1)}_{fi} = -ig\int \chi^*_f(t,\mathbf{x})\bar{\psi}_f(t,\mathbf{x})\psi_i(t,\mathbf{x})d^3x dt, \tag{84}$$

where ψ_i and ψ_f are the wave functions of the initial and final fermion, respectively, and χ_f is the wave function of the final meson. From Eqs. (82) and (84), we obtain the expression for the first-order decay amplitude $T^{(1)}_{fi}$

$$T^{(1)}_{fi} = -g\int \chi^*_f(t,\mathbf{x})\bar{\psi}_f(t,\mathbf{x})\psi_i(t,\mathbf{x})dx. \tag{85}$$

Note that in Eq. (85), the integration is performed over the x-coordinate only, while the coordinates y, z, and time t are set equal to zero. Now, let us consider three decay channels (83) separately.

1. Decay channel $f_1 \to f_0 + \chi_0$

This decay channel corresponds to the transition of the fermion from the first excited state to the massless state with the emission of the massless meson. Let us denote the four-momenta of the initial fermion, final fermion, and final meson by p, p', and k', respectively. Then we have in the rest frame of the initial fermion

$$p = (\epsilon_1, \mathbf{0}), \qquad p' = (\epsilon', \mathbf{p}'_{\parallel}), \qquad k' = (\omega', -\mathbf{p}'_{\parallel}), \tag{86}$$

where the particle energies are

$$\epsilon_1 = \frac{\sqrt{2\nu-1}}{w}, \qquad \epsilon' = \omega' = \frac{\sqrt{2\nu-1}}{2w}, \tag{87}$$

and the absolute values of the final-state momenta $\mathbf{p}'_{\parallel}$ and $\mathbf{k}'_{\parallel} = -\mathbf{p}'_{\parallel}$ are equal to ϵ'. Note that the decay $f_1 \to f_0 + \chi_0$ is kinematically two dimensional, because all particles are localized on the domain wall. Substituting Eqs. (9), (59), and (61) for the wave functions in Eq. (85), we obtain the analytical expression of the decay amplitude

$$T_{\frac{1}{2}} = (T_{-\frac{1}{2}})^* = g\sqrt{\frac{2}{w\epsilon'}}R(\nu)\exp\left(i\frac{\varphi}{2}\right), \tag{88}$$

where φ is the azimuthal angle of the massless fermion, and the factor $R(\nu)$ is expressed in terms of beta functions:

$$R(\nu) = \frac{\sqrt{3}}{8}\left(\frac{2\nu-2}{2\nu-1}\right)^{1/2}\frac{\mathrm{B}(\frac{1}{2},\nu+\frac{1}{2})}{\mathrm{B}(\frac{1}{2},\nu)}. \tag{89}$$

The superscripts $\pm 1/2$ in Eq. (88) indicate the polarization states $s_x = \pm 1/2$ of the initial fermion that is in the first excited state. From Eqs. (88) and (89), we obtain the analytical expressions for the differential and total decay probabilities per unit time:

$$\frac{d\Gamma}{d\varphi} = \frac{g^2}{2\pi w}|R(\nu)|^2, \tag{90}$$

$$\Gamma = \frac{g^2}{w}|R(\nu)|^2. \tag{91}$$

From Eq. (90), it follows that the angular distribution of the products of the decay $f_1 \to f_0 + \chi_0$ is isotropic in the rest frame of the initial fermion. From Eqs. (89) and (91), it is possible to obtain the series expansions of the decay width Γ for the limiting regimes $\nu \to 1$ and $\nu \to \infty$. For $\nu \to 1$, we have

$$\Gamma = \alpha_1(\nu - 1) + \alpha_2(\nu - 1)^2 + O((\nu - 1)^3), \qquad (92)$$

where the series coefficients α_1 and α_2 are

$$\alpha_1 = \frac{3}{512}\pi^2 g^2 w^{-1}, \qquad (93)$$

$$\alpha_2 = \frac{3}{128}\pi^2 g^2 w^{-1}\left(\gamma - 1 + \psi\left(\frac{3}{2}\right)\right). \qquad (94)$$

In Eq. (94), $\gamma \approx 0.577216$ is the Euler-Mascheroni constant, and $\psi(3/2) \approx 0.03649$ is the value of the digamma function at $3/2$. For $\nu \to \infty$, we obtain the following asymptotic expression for the decay width Γ:

$$\Gamma \sim \frac{g^2}{w}\left(\frac{3}{64} - \frac{3}{64}\frac{1}{\nu} + \frac{3}{512}\frac{1}{\nu^2} + O\left(\frac{1}{\nu^3}\right)\right). \qquad (95)$$

2. Decay channel $f_1 \to f_0 + \chi_1$

It can be shown that in this case, the first-order decay amplitude $T_{fi}^{(1)}$ is equal to zero. Indeed, the first-order amplitude for the decay $f_1 \to f_0 + \chi_1$ is

$$T_{fi}^{(1)} = -g\int \chi_1^*(x, \mathbf{k}'_\parallel)\bar{\Psi}_0(x, \mathbf{p}'_\parallel)\Psi_{s_x,1}(x, \mathbf{p}_\parallel)dx, \qquad (96)$$

where only the dependence on x is shown, while the y- and z-coordinates are assumed to be equal to zero. Let the initial massive fermion be at rest ($\mathbf{p}_\parallel = 0$, $\mathbf{p}'_\parallel = -\mathbf{k}'_\parallel$); then, from Eq. (81), it follows that the initial fermionic state with the spin x-projection s_x has the P-parity equal to $(-1)^{s_x+1/2}$. From Eq. (81), it also follows that $P\chi_1(x, \mathbf{k}'_\parallel) = \chi_1(x, \mathbf{k}'_\parallel)$; i.e., the wave function of the excited bosonic mode at $y = 0$, $z = 0$ is the eigenfunction of transformation (80) with the eigenvalue equal to 1. The wave function $\Psi_0(x, \mathbf{p}'_\parallel)$ at $y = 0$, $z = 0$ is not an eigenfunction of transformation (80), but it can be shown that

$$\Psi_0(x, \mathbf{p}'_\parallel) = \Psi_0^{-1}(x, \mathbf{p}'_\parallel) + \Psi_0^{1}(x, \mathbf{p}'_\parallel), \qquad (97)$$

where

$$P\Psi_0^{\pm 1}(x, \mathbf{p}'_\parallel) = \pm\Psi_0^{\pm 1}(x, \mathbf{p}'_\parallel),$$
$$s_x\Psi_0^{\pm 1}(x, \mathbf{p}'_\parallel) = \pm\frac{1}{2}\Psi_0^{\pm 1}(x, \mathbf{p}'_\parallel). \qquad (98)$$

Furthermore, the structure of the Yukawa interaction in Eq. (96) is such that spin flip transitions are forbidden, because $\bar{\Psi}_0^{-1}(x, \mathbf{p}'_\parallel)\Psi_{\frac{1}{2},1}(x, \mathbf{p}'_\parallel)$ and $\bar{\Psi}_0^{1}(x, \mathbf{p}'_\parallel)\Psi_{-\frac{1}{2},1}(x, \mathbf{p}'_\parallel)$ vanish identically. Putting all this together, we see that the first-order decay amplitude (96) changes the sign under transformation (80), and so it must be equal to zero. Mathematically, this means that the integrand of Eq. (96) is an odd function of x; hence, the integral in Eq. (96) vanishes.

3. Decay channel $f_1 \to f_0 + \chi_{\mathrm{k}}$

The characteristic feature of the decay channel $f_1 \to f_0 + \chi_{\mathrm{k}}$ is that the final meson is not localized on the domain wall. Correspondingly, the three-momentum $\mathbf{k}'$ of the final meson has a component k'_x that is perpendicular to the domain wall's plane. Let us choose the angle θ between the meson momentum $\mathbf{k}'$ and the normal to the domain wall's plane as the independent kinematic variable. We denote the four-momenta of the initial fermion, final fermion, and final meson by p, p', and k', respectively. Then we have in the rest frame of the initial fermion

$$p = (\epsilon_1, \mathbf{0}), \qquad p' = (\epsilon', -\mathbf{k}'_\parallel), \qquad k' = (\omega', k'_x, \mathbf{k}'_\parallel), \qquad (99)$$

where

$$k'_x = w^{-1}\left(\sqrt{2\nu - 1 - 4\cos^2(\theta)}\sec(\theta) - \sqrt{2\nu - 1}\tan(\theta)\right), \qquad (100)$$

$$\epsilon' = |\mathbf{k}'_\parallel| = \frac{2\nu - 5 - k_x'^2 w^2}{2w\sqrt{2\nu - 1}}, \qquad (101)$$

$$\omega' = \frac{3 + 2\nu + k_x'^2 w^2}{2w\sqrt{2\nu - 1}}. \qquad (102)$$

Note that the x-projection of the total three-momentum is not conserved in the decay $f_1 \to f_0 + \chi_{\mathrm{k}}$. Substituting Eqs. (11), (59), and (61) for the wave functions in Eq. (85), we obtain the analytical expression for the amplitude of the decay $f_1 \to f_0 + \chi_{\mathrm{k}}$,

$$T_{\frac{1}{2}} = (T_{-\frac{1}{2}})^* = i\sqrt{2}g\exp\left(i\frac{\varphi}{2}\right)\frac{1}{\sqrt{2\omega'}}\sqrt{\frac{1 + k_x'^2 w^2}{4 + k_x'^2 w^2}}R(\nu, k'_x), \qquad (103)$$

where φ is the azimuthal angle of the final meson in the domain wall's plane, and the factor $R(\nu, k'_x)$ can be written compactly in terms of beta functions:

$$R(\nu, k_x') = 2^{2(\nu-3)}(3+4\nu(\nu-2)) \times \sqrt{\frac{\nu-1}{2\nu-1}} \mathrm{B}\left(\frac{3}{2}, \nu-\frac{1}{2}\right) \times \mathrm{B}\left(\nu-\frac{1}{2}-i\frac{k_x' w}{2}, \nu-\frac{1}{2}+i\frac{k_x' w}{2}\right). \quad (104)$$

From Eqs. (103) and (104), we obtain expressions for the differential probabilities of the decay $f_1 \to f_0 + \chi_\mathrm{k}$ per unit time:

$$\frac{d\Gamma}{dk_x' d\varphi} = \frac{g^2}{8\pi^2} \frac{(1+k_x'^2 w^2)(2\nu-5-k_x'^2 w^2)}{(4+k_x'^2 w^2)(2\nu-1)} \times |R(\nu, k_x')|^2, \quad (105)$$

$$\frac{d\Gamma}{d\theta d\varphi} = \frac{d\Gamma}{dk_x' d\varphi}\left|\frac{dk_x'}{d\theta}\right|, \quad (106)$$

where

$$\frac{dk_x'}{d\theta} = -\frac{\epsilon_1 k_x'}{k_x' + (\omega'/2)\sin(2\theta)}. \quad (107)$$

From Eqs. (104)–(107), it follows that $d\Gamma/dk_x' d\varphi$ is invariant under the change $k_x' \to -k_x'$ and that $d\Gamma/d\theta d\varphi$ is invariant under the change $\theta \to \pi - \theta$. Thus, the angular distribution of the final meson χ_k is invariant under the reflection about the domain wall's plane. This is because the Lagrangian (77) is invariant under parity transformation (80) and under a rotation about the x-axis.

Figure 1 shows the dependence of the dimensionless combination $wg^{-2}d\Gamma/d\theta d\varphi$ on the polar angle θ of the final meson χ_k for several values of the parameter ν. The dependence is presented in the polar coordinates $\rho = wg^{-2}d\Gamma/d\theta d\varphi$ and θ. The right (left) parts of the curves in Fig. 1 correspond to the azimuthal angle φ $(\varphi+\pi)$, where the value of φ can be chosen arbitrarily because of the azimuthal isotropy of $d\Gamma/d\theta d\varphi$. From Fig. 1, it follows that $d\Gamma/d\theta d\varphi$ increases rapidly for all values of θ as the parameter ν increases, with the exception of $\theta = 0$, π at which $d\Gamma/d\theta d\varphi$ vanishes. This is because $d\Gamma/d\theta d\varphi$ contains the factor $|\mathbf{k}_\parallel'|$ that vanishes at $\theta = 0$, π. Note that in line with the above, $d\Gamma/d\theta d\varphi$ is even under the reflection about the domain wall's plane.

Unfortunately, the decay width $\Gamma = \int (d\Gamma/d\theta d\varphi) d\theta d\varphi$ cannot be calculated analytically for an arbitrary ν. Instead, it is possible to obtain the series expansions of $\Gamma(\nu)$ with respect to ν for two asymptotic regimes. For $\delta = \nu - 5/2 \to 0$, we have the following series expansion:

$$\Gamma = \frac{g^2}{w}\frac{16\sqrt{2}}{2025\pi^3}\delta^{\frac{3}{2}}(1+\alpha_1\delta+\alpha_2\delta^2+O(\delta^3)), \quad (108)$$

where the series coefficients α_1 and α_2 are

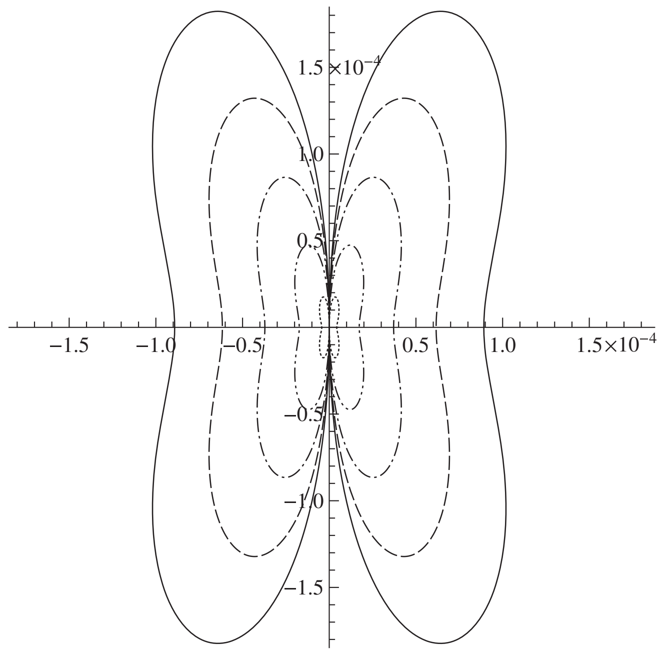

FIG. 1. Dependence of the dimensionless combination $wg^{-2}d\Gamma/d\theta d\varphi$ on the polar angle θ of the final meson χ_k. The dependence is presented in polar coordinates $\rho = wg^{-2}d\Gamma/d\theta d\varphi$ and θ. The dotted, dash-dot-dotted, dash-dotted, dashed, and solid curves correspond to the parameter $\nu = 3$, 3.5, 4, 4.5, and 5, respectively.

$$\alpha_1 = -\frac{43}{10} - \frac{\pi^2}{30} + \ln(256) \approx 0.916191, \quad (109)$$

$$\alpha_2 = \frac{2\zeta(3)}{5} + \frac{30683 - 1109\pi^2}{2100} + 8\ln^2(4) - \frac{2}{15}(129+\pi^2)\ln(4) \approx -0.414373. \quad (110)$$

For $\nu \to \infty$, the asymptotic expansion of the decay width Γ has the form

$$\Gamma \sim \frac{g^2}{w}(\beta_{\frac{1}{2}}\nu^{\frac{1}{2}} + \beta_0 + \beta_{-\frac{1}{2}}\nu^{-\frac{1}{2}} + \beta_{-1}\nu^{-1} + O(\nu^{-\frac{3}{2}})), \quad (111)$$

where the expansion coefficients are

$$\beta_{\frac{1}{2}} = \frac{59}{\pi\sqrt{2840}} \approx 0.0158091,$$
$$\beta_0 = -\beta_{-1} = -\frac{3}{64} \approx -0.046875,$$
$$\beta_{-\frac{1}{2}} = \frac{89}{\pi\sqrt{21680}} \approx 0.0119238. \quad (112)$$

Figure 2 presents the dependence of the dimensionless combination $w\Gamma/g^2$ on the parameter ν. The dependence was obtained numerically. From Fig. 2, it follows that $\Gamma(\nu)$ is the monotonically increasing function of ν. Note that the

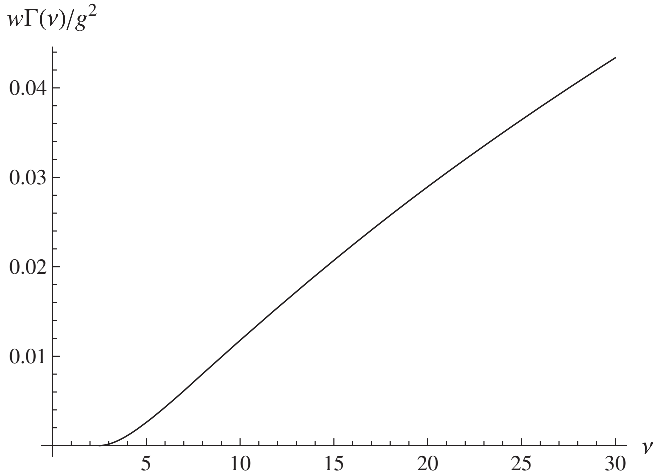

FIG. 2. Dependence of the dimensionless combination $w\Gamma/g^2$ on the parameter ν for the decay $f_1 \to f_0 + \chi_{\mathrm{k}}$.

behavior of $\Gamma(\nu)$ near the left boundary point $\nu = 5/2$ and when $\nu \gg 1$ corresponds to Eqs. (108) and (111), respectively.

B. Decays of the excited bosonic mode

From Eq. (77), it follows that a decay of the excited bosonic mode can be either bosonic or fermionic. The bosonic decays are

$$\begin{aligned} &\chi_1 \to \chi_0 + \chi_0, \\ &\chi_1 \to \chi_0 + \chi_0 + \chi_0, \end{aligned} \tag{113}$$

where χ_0 and χ_1 are mesons of the massless and excited modes, respectively. The bosonic decays are kinematically allowable for any values of the model parameters. The fermionic decays are of the annihilation type $\chi_1 \to f_a + \bar{f}_b$, where f_a and $\bar{f}_b$ are some fermion and antifermion, respectively, in a final state. It is easily shown that the kinematically allowable fermionic decays of the excited bosonic mode are only

$$\begin{aligned} &\chi_1 \to f_0 + \bar{f}_0 \quad \text{for } \nu > 0, \\ &\chi_1 \to f_0 + \bar{f}_1, \qquad \chi_1 \to \bar{f}_0 + f_1 \quad \text{for } \nu \in (1, 2), \\ &\chi_1 \to f_0 + \bar{f}_{\mathrm{p}}, \qquad \chi_1 \to \bar{f}_0 + f_{\mathrm{p}} \quad \text{for } \nu \in (0, \sqrt{3}), \\ &\chi_1 \to f_{\mathrm{p}} + \bar{f}_{\mathrm{p}} \quad \text{for } \nu \in (0, \sqrt{3}/2), \end{aligned} \tag{114}$$

where f_{p} ($\bar{f}_{\mathrm{p}}$) denotes the nonlocalized fermion (antifermion). All other fermionic channels of the decay are forbidden kinematically. Now, let us consider decay channels (113) and (114). Since the Lagrangian (77) is invariant under the charge conjugation, we consider only one decay for each of the two charge-conjugate pairs of decays in Eq. (114).

1. *Decay channel* $\chi_1 \to \chi_0 + \chi_0$

We denote the four-momenta of the initial meson and the final mesons by k, k', and k'', respectively. Then we have in the center-of-mass frame

$$k = (\omega, \mathbf{0}), \qquad k' = (\omega', \mathbf{k}'_{\parallel}), \qquad k'' = (\omega'', -\mathbf{k}'_{\parallel}), \tag{115}$$

where the energies of the particles are

$$\omega = \frac{\sqrt{3}}{w}, \qquad \omega' = \omega'' = \frac{\sqrt{3}}{2w}, \tag{116}$$

and the absolute values of the final-state momenta $\mathbf{k}'_{\parallel}$ and $\mathbf{k}''_{\parallel} = -\mathbf{k}'_{\parallel}$ are equal to ω'. The expression of the first-order amplitude for the decay is

$$\begin{aligned} T_{fi}^{(1)} &= -6\lambda \int \phi_{\mathrm{w}}(x)\chi_0^*(x, \mathbf{k}')\chi_0^*(x, -\mathbf{k}')\chi_1(x, \mathbf{0})dx \\ &= -\frac{3^{7/4}\pi}{32}\frac{1}{w\eta}, \end{aligned} \tag{117}$$

where we use Eqs. (7), (9), and (10) for the wall solution ϕ_{w}, the wave function of the massless bosonic mode χ_0, and the wave function of the excited bosonic mode χ_1, respectively. From Eq. (117), we obtain expressions for the differential and total decay probabilities per unit time:

$$\frac{d\Gamma}{d\varphi} = \frac{81\pi}{8192}\frac{1}{w^3\eta^2}, \tag{118}$$

$$\Gamma = \frac{1}{2}\frac{81\pi^2}{4096}\frac{1}{w^3\eta^2}, \tag{119}$$

where φ is the azimuth angle of the massless meson. The factor $1/2$ in Eq. (119) takes into account the identity of the two final mesons.

2. *Decay channel* $\chi_1 \to \chi_0 + \chi_0 + \chi_0$

It is easily shown that the amplitude of this decay vanishes to the first order in the coupling constant λ. Indeed, the expression of the first-order amplitude is

$$T_{fi}^{(1)} = -6\lambda \int \chi_0^*(x, \mathbf{k}')\chi_0^*(x, \mathbf{k}'')\chi_0^*(x, \mathbf{k}''')\chi_1(x, \mathbf{0})dx. \tag{120}$$

From Eqs. (9), (10), and (81), it follows that the first-order decay amplitude (120) changes the sign under transformation (80); hence, it must be equal to zero. Indeed, the integrand of Eq. (120) is an odd function of x, so integral (120) vanishes.

3. Decay channel $\chi_1 \to f_0 + \bar{f}_0$

The first-order amplitude of the decay $\chi_1 \to f_0 + \bar{f}_0$ is written as

$$T_{fi}^{(1)} = -g \int \bar{\Psi}_0(x, \mathbf{p}'_\parallel)\Psi_0^c(x, -\mathbf{p}''_\parallel)\chi_1(x, \mathbf{0})dx, \qquad (121)$$

where $\mathbf{p}'_\parallel$ and $\mathbf{p}''_\parallel = -\mathbf{p}'_\parallel$ are the momenta of the final fermion and antifermion in the center-of-mass frame. However, it follows from Eq. (64) that the product $\bar{\Psi}_0(x, \mathbf{p}'_\parallel)\Psi_0^c(x, -\mathbf{p}''_\parallel)$ vanishes identically, and so the first-order amplitude (121) does as well.

4. Decay channel $\chi_1 \to f_0 + \bar{f}_1$

This channel is connected to the channel $f_1 \to f_0 + \chi_1$ by the crossing transformation. It was shown in Sec. IV A 2 that the first-order amplitude of the decay $f_1 \to f_0 + \chi_1$ vanishes, because the amplitude changes the sign under parity transformation (80). Similarly, it can be shown that the first-order amplitude of the decay $\chi_1 \to f_0 + \bar{f}_1$ changes the sign under parity transformation (80); hence, it must be equal to zero. Indeed, the corresponding integrand is an odd function of x again, and the first-order amplitude of the decay $\chi_1 \to f_0 + \bar{f}_1$ vanishes.

5. Decay channel $\chi_1 \to \bar{f}_0 + f_\mathrm{p}$

In this channel, the final massive fermion is not localized on the domain wall, and its three-momentum has a component that is perpendicular to the domain wall's plane. We denote the four-momenta of the initial meson, the final fermion, and the final antifermion by k, p', and p'', respectively, and choose the angle θ between the fermion three-momentum $\mathbf{p}'$ and the normal to the domain wall's plane as the independent kinematic variable. Then, we have in the rest frame of the initial meson

$$k = (\sqrt{3}/w, \mathbf{0}), \qquad p' = (\epsilon', p'_x, \mathbf{p}'_\parallel),$$
$$p'' = (\epsilon'', -\mathbf{p}'_\parallel), \qquad (122)$$

where

$$p'_x = w^{-1}\left(\sqrt{3 - \nu^2\cos^2(\theta)}\sec(\theta) - \sqrt{3}\tan(\theta)\right),$$
$$\epsilon' = \frac{3 + \nu^2 + \mu^2}{2\sqrt{3}w}, \qquad \epsilon'' = |\mathbf{p}'_\parallel| = \frac{3 - \nu^2 - \mu^2}{2\sqrt{3}w}, \qquad (123)$$

and $\mu = w|p'_x|$. From Eq. (123), it follows that the kinematically allowable domain of the parameter ν is $(0, \sqrt{3})$.

Let us discuss the question about the wave functions of the final nonlocalized fermion f_p, which must be used for the calculation of the decay amplitudes. In this connection, we must remember the bremsstrahlung and the pair creation in a Coulomb field of a heavy nucleus. It is known [43] that at large distances from the nucleus, wave functions of final fermions (electrons) must be the superposition of a plane wave and an *ingoing* spherical wave, where the amplitude of the plane wave must be normalized so that the number of electrons per unit volume is equal to unity. At the same time, the wave functions of the nonlocalized fermionic modes (Sec. III B 1) are the superposition of the one (incident) plane wave moving to the domain wall and the two (transmitted and reflected) plane waves moving *from* the domain wall. Therefore, in our case, the wave function of the final fermion f_p must be the superposition of the two plane waves (reversed transmitted and reversed reflected) moving *to* the domain wall and the one plane wave (reversed incident) moving from the domain wall. It can be shown that these reversed wave functions are obtained from those of Sec. III B 1 by changing $\mu \to -\mu$ (which is equivalent to changing $|p'_x| \to -|p'_x|$, because $\mu = w|p'_x|$). From Eqs. (40) and (41), it follows that these reversed wave functions are normalized so that the number of the fermions moving from the domain wall per unit volume is equal to unity at large distances, just as it should be.

The first-order amplitude of the decay $\chi_1 \to \bar{f}_0 + f_\mathrm{p}$ is written as

$$T_{s_x}^d = -g \int \bar{\Psi}_{s_x}^d(x, -|p'_x|, \mathbf{p}'_\parallel)\Psi_0^c(x, -\mathbf{p}'_\parallel)\chi_1(x, \mathbf{0})dx, \qquad (124)$$

where the index d is l or r, and $s_x = \pm 1/2$ defines the spin state of the final fermion f_p. Substituting Eqs. (10), (38), and (62) for the wave functions in Eq. (124), we obtain expressions for the decay amplitudes:

$$T_{s_x}^l = T_{s_x}^r = N(\nu, \mu)F(\nu, \mu)\exp(-is_x\varphi), \qquad (125)$$

where φ is the azimuthal angle of the momentum $\mathbf{p}'_\parallel$ lying in the domain wall's plane. Note that the index l (r) in Eq. (125) now corresponds to the outgoing plane wave that moved to the right (left) *from* the domain wall. The factor $N(\nu, \mu)$ in Eq. (125) can be written analytically as

$$N(\nu, \mu) = -\frac{g}{4}\left[\frac{\sqrt{3}\pi}{w\mathrm{B}(1/2, \nu)}\frac{\mu\mathcal{T}(\mu, \nu)}{\sinh(\pi\mu)}(1 + v_\parallel)\right]^{\frac{1}{2}}$$
$$\times \exp[i\kappa(\nu, \mu)], \qquad (126)$$

where $\mathcal{T}(\mu, \nu)$ is the transmission coefficient (43), $v_\parallel = |\mathbf{p}'_\parallel|/\epsilon'$ is the component of the velocity of the final fermion lying in the domain wall's plane, and the phase $\kappa(\nu, \mu)$ is

$$\kappa(\nu, \mu) = \arctan(\nu/\mu) - 2\arg[\Gamma(\nu + i\mu)]. \qquad (127)$$

The form factor $F(\nu, \mu)$ is expressed by the following integral:

$$F(\nu,\mu) = w\int_{-\infty}^{\infty} \operatorname{sech}^{2+\nu}(\xi)\sinh(\xi)P_{\nu-1}^{i\mu}(\tanh(\xi))d\xi. \tag{128}$$

This integral cannot be evaluated analytically for arbitrary values of ν and μ, but using properties of the associated Legendre functions [37], we can determine some general properties of the form factor F:

$$F(\nu,-\mu) = F(\nu,\mu)^* \Rightarrow \mathrm{Im}[F(\nu,0)] = 0,$$
$$F(0,0) = F(1,0) = 0. \tag{129}$$

When the parameter ν tends to the limiting value $\sqrt{3}$, the form factor F also tends to the real limiting value:

$$F(\nu,\mu) \underset{\nu\to\sqrt{3}}{\longrightarrow} \mathrm{const} \approx 0.416833w. \tag{130}$$

Using Eqs. (125), (126), and (128), we obtain expressions for the differential probabilities of the decay per unit time:

$$\frac{d\Gamma}{dp'_x d\varphi} = \frac{2}{(2\pi)^2}|N(\nu,\mu)|^2|F(\nu,\mu)|^2\frac{w\epsilon'\epsilon''}{\sqrt{3}}, \tag{131}$$

$$\frac{d\Gamma}{d\theta d\varphi} = \frac{d\Gamma}{dp'_x d\varphi}\left|\frac{dp'_x}{d\theta}\right|. \tag{132}$$

The factor of 2 in Eq. (131) arises because the two amplitudes $T^l_{\pm 1/2}$ contribute to decays for which $p'_x > 0$, while the two other amplitudes $T^r_{\pm 1/2}$ contribute to decays for which $p'_x < 0$. The kinematic factors of Eqs. (131) and (132) can be expressed compactly in terms of the parameters ν and μ:

$$\epsilon'\epsilon'' = \frac{9-(\nu^2+\mu^2)}{12w^2},$$
$$\frac{dp'_x}{d\theta} = -\frac{\mu^4+(\nu^2-3)^2+2\mu^2(\nu^2+3)}{2\sqrt{3}w(3+\mu^2-\nu^2)}. \tag{133}$$

From Eqs. (126)–(129), it follows that $d\Gamma/dp'_x d\varphi$ is invariant under the change $p'_x \to -p'_x$, and that $d\Gamma/d\theta d\varphi$ is invariant under the change $\theta \to \pi - \theta$. Thus, the angular distribution of the final fermion f_{p} is invariant under the reflection about the domain wall. This fact is the consequence of the invariance of Lagrangian (77) under parity transformation (80) and under a rotation about the x-axis.

In Fig. 3, we can see the dependence of the dimensionless combination $wg^{-2}d\Gamma/d\theta d\varphi$ on the fermion polar angle θ. The dependence is presented in polar coordinates $\rho = wg^{-2}d\Gamma/d\theta d\varphi$ and θ, as it is in Fig. 1. The curves in Fig. 3 are symmetric under the reflection about the domain wall's plane, as with those in Fig. 1. Yet unlike the curves in Fig. 1, the curves in Fig. 3 pass through the origin of

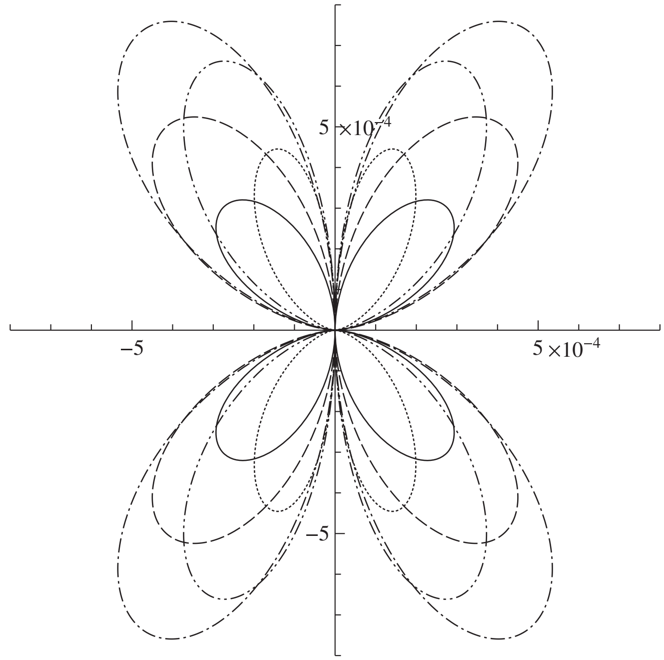

FIG. 3. Dependence of the dimensionless combination $wg^{-2}d\Gamma/d\theta d\varphi$ on the polar angle θ of the final fermion f_{p}. The dependence is presented in polar coordinates $\rho = wg^{-2}d\Gamma/d\theta d\varphi$ and θ. The solid, dashed, dash-dotted, dash-dot-dotted, and dotted curves correspond to the parameter $\nu = 0.075$, 0.15, 0.4, 0.7, and 1, respectively.

the coordinates not only at $\theta = 0$, π but also at $\theta = \pi/2$. Thus, the three-momentum of the final fermion f_{p} cannot lie in the domain wall's plane. Note in this connection that Eq. (131) for $d\Gamma/dp'_x d\varphi$ contains the factors $\epsilon'' = |\mathbf{p}'_{\parallel}|$, $\mathcal{T}(\mu,\nu)$, and $|F(\nu,\mu)|^2$. The reason for the vanishing of $d\Gamma/dp'_x d\varphi$ at $\theta = 0, \pi$ is that the factor $|\mathbf{p}'_{\parallel}|$ also vanishes at $\theta = 0$, π. Further, when the polar angle θ is equal to $\pi/2$, the parameter $\mu = w|p'_x|$ is equal to zero. If $\mu = 0$ and $\nu \neq 1$, then the transition coefficient $\mathcal{T}(0,\nu)$ vanishes. If $\mu = 0$ and $\nu = 1$, then from Eq. (129) it follows that the form factor $F(1,0)$ vanishes. Thus, we conclude that $d\Gamma/dp'_x d\varphi$ must vanish at $\theta = \pi/2$.

The decay width $\Gamma = \int(d\Gamma/d\theta d\phi)d\theta d\phi$ cannot be calculated analytically for any ν, but it can be obtained numerically. Figure 4 shows the dependence of the dimensionless combination $w\Gamma/g^2$ on the parameter ν in the kinematically allowable domain $(0,\sqrt{3})$. From Fig. 4, it follows that $\Gamma(\nu)$ vanishes as ν tends to the boundary points 0 or $\sqrt{3}$. In particular, it is found numerically that

$$wg^{-2}\Gamma(\nu) \approx 0.046\nu \quad \text{as} \quad \nu \to 0 \tag{134}$$

and

$$wg^{-2}\Gamma(\nu) \approx 0.043(\nu-\sqrt{3})^{\frac{5}{2}} \quad \text{as} \quad \nu \to \sqrt{3}. \tag{135}$$

Note that $\Gamma(\nu)$ vanishes as $\nu \to 0$, in spite of the fact that the phase volume of the decay $\chi_1 \to \bar{f}_0 + f_{\mathrm{p}}$ reaches a

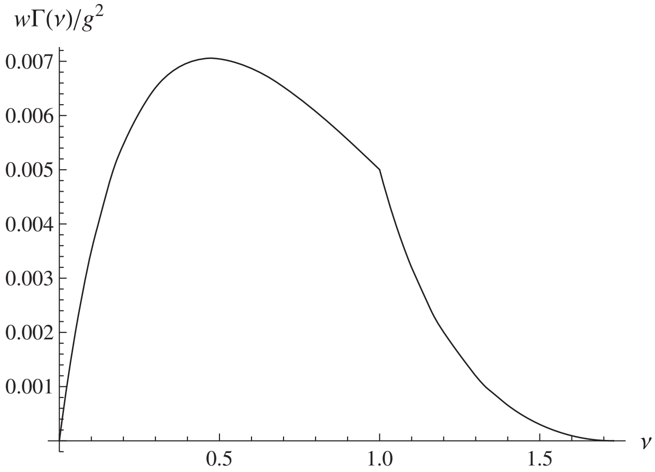

FIG. 4. Dependence of the dimensionless combination $w\Gamma/g^2$ on the parameter ν for the decay $\chi_1 \to \bar{f}_0 + f_{\mathrm{p}}$.

maximum value at this point. This is because the factor $|N(\nu,\mu)|^2$ in Eq. (131) contains the factor $\mathrm{B}(1/2,\nu)^{-1}$ that vanishes as ν tends to zero. The factor $\mathrm{B}(1/2,\nu)^{-1}$ arises from the massless mode's normalization constant $\mathcal{N}_0$ [Eq. (70)] that also vanishes as ν tends to zero. This is because the massless fermionic and antifermionic modes (61) and (62) spread over the x-axis as the parameter ν tends to zero. Another characteristic feature of the dependence in Fig. 4 is the presence of the cusp at $\nu = 1$. This is because the transmission coefficient $\mathcal{T}(\mu,\nu)$ in Eq. (126) has nonregular behavior [Eqs. (46)–(49)] as $\nu \to 1$, $\mu \to 0$.

6. Decay channel $\chi_1 \to f_{\mathrm{p}} + \bar{f}_{\mathrm{p}}$

In this channel, both final particles are not localized on the domain wall. We denote the four-momenta of the initial meson, the final fermion, and the final antifermion by k, p', and p'', respectively, and choose the perpendicular components p'_x and p''_x as the independent kinematic variables. Then, we have in the rest frame of the initial meson

$$k = (\sqrt{3}/w, \mathbf{0}), \qquad p' = (\epsilon', p'_x, \mathbf{p}'_{\parallel}),$$
$$p'' = (\epsilon'', p''_x, -\mathbf{p}'_{\parallel}), \tag{136}$$

where

$$|\mathbf{p}'_{\parallel}| = \frac{1}{2\sqrt{3}w}[9 - 6(\mu'^2 + \mu''^2) + (\mu'^2 - \mu''^2)^2 - 12\nu^2]^{\frac{1}{2}}, \tag{137}$$

$\mu' = w|p'_x|$, and $\mu'' = w|p''_x|$. From Eq. (137), it follows that the kinematically allowable domain of the parameter ν is $(0, \sqrt{3}/2)$.

The first-order amplitude of the decay $\chi_1 \to f_{\mathrm{p}} + \bar{f}_{\mathrm{p}}$ is written as

$$T^{d'd''}_{s'_x s''_x} = -g\int \bar{\Psi}^{d'}_{s'_x}(x, -|p'_x|, \mathbf{p}'_{\parallel}) \times \Psi^{d''c}_{s''_x}(x, -|p''_x|, -\mathbf{p}'_{\parallel})\chi_1(x, \mathbf{0})dx. \tag{138}$$

Substituting Eqs. (10), (38), and (53) for the wave functions in Eq. (138), we obtain expressions for the decay amplitudes:

$$T^{ll}_{s'_x s''_x} = T^{rr}_{s'_x s''_x} = (-1)^{\frac{|s'_x + s''_x - 1|}{2}} \exp(-i(s'_x + s''_x)\varphi') \times \frac{\tau'\tau'' + (-1)^{s'_x + s''_x}}{\sqrt{(1+\tau'^2)(1+\tau''^2)}} N(\nu, \mu', \mu'') \times [\exp(i\arctan(\nu/\mu'))F(\nu, \mu'', \mu') + (-1)^{s'_x + s''_x + 1}\{\mu' \leftrightarrow \mu''\}], \tag{139}$$

$$T^{rl}_{s'_x s''_x} = T^{lr}_{s'_x s''_x} = (-1)^{\frac{|s'_x + s''_x - 1|}{2}} \exp(-i(s'_x + s''_x)\varphi') \times \frac{\tau'\tau'' + (-1)^{s'_x + s''_x}}{\sqrt{(1+\tau'^2)(1+\tau''^2)}} N(\nu, \mu', \mu'') \times [\exp(i\arctan(\nu/\mu'))H(\nu, \mu'', \mu') + (-1)^{s'_x + s''_x + 1}\{\mu' \leftrightarrow \mu''\}], \tag{140}$$

where φ' is the azimuthal angle of the momentum $\mathbf{p}'_{\parallel}$, the factor $N(\nu, \mu', \mu'')$ is

$$N(\nu, \mu', \mu'') = 3^{\frac{1}{4}}\pi\frac{g}{4}\left[\frac{\mu'\mathcal{T}(\mu', \nu)\mu''\mathcal{T}(\mu'', \nu)}{\sinh(\pi\mu')\sinh(\pi\mu'')}\right]^{\frac{1}{2}}, \tag{141}$$

and the parameters τ' and τ'' are defined in terms of the velocities $v'_{\parallel} = |\mathbf{p}'_{\parallel}|/\epsilon'$ and $v''_{\parallel} = |\mathbf{p}'_{\parallel}|/\epsilon''$:

$$\tau' = \sqrt{\frac{1 + v'_{\parallel}}{1 - v'_{\parallel}}}, \qquad \tau'' = \sqrt{\frac{1 + v''_{\parallel}}{1 - v''_{\parallel}}}. \tag{142}$$

The form factors $F(\nu, \mu'', \mu')$ and $H(\nu, \mu'', \mu')$ in Eqs. (139) and (140) can be written as

$$F(\nu, \mu', \mu'') = w\int_{-\infty}^{\infty}\sinh(\xi)\mathrm{sech}^2(\xi) \times f(\nu, \mu', \xi)f(\nu - 1, \mu'', \xi)d\xi, \tag{143}$$

$$H(\nu, \mu', \mu'') = w\int_{-\infty}^{\infty}\sinh(\xi)\mathrm{sech}^2(\xi) \times f(\nu, \mu', \xi)h(\nu - 1, \mu'', \xi)d\xi, \tag{144}$$

where the functions $f(a, b, \xi)$ and $h(a, b, \xi)$ are

$$f(a,b,\xi) = \cos(\pi a)\left[P_a^{-ib}(\tanh(\xi)) - \frac{2}{\pi}\tan(\pi a)Q_a^{-ib}(\tanh(\xi))\right], \quad (145)$$

$$h(a,b,\xi) = \cosh(\pi b)\left[P_a^{-ib}(\tanh(\xi)) + \frac{2i}{\pi}\tanh(\pi b)Q_a^{-ib}(\tanh(\xi))\right]. \quad (146)$$

Some general properties of the form factors F and H are

$$F(\nu,-\mu',-\mu'') = F(\nu,\mu',\mu'')^*,$$
$$H(\nu,-\mu',-\mu'') = H(\nu,\mu',\mu'')^*,$$
$$F(0,0,0) = H(0,0,0) = 0. \quad (147)$$

When the parameter ν tends to the limiting value $\sqrt{3}/2$, the form factors F and H also tend to their real limiting values:

$$F(\nu,\mu',\mu'') \underset{\nu\to\sqrt{3}/2}{\longrightarrow} \text{const} \approx 2.18859w,$$
$$H(\nu,\mu',\mu'') \underset{\nu\to\sqrt{3}/2}{\longrightarrow} \text{const} \approx 2.08196w. \quad (148)$$

From Eqs. (139) and (140), it follows that the decay amplitudes satisfy the following symmetry relations:

$$T^{d'd''}_{s'_x s''_x}[\mu',\mu'',\varphi'] = (-1)^{s'_x+s''_x} T^{\bar{d}'\bar{d}''}_{s'_x s''_x}[\mu',\mu'',\varphi'+\pi], \quad (149)$$

$$T^{d'd''}_{s'_x s''_x}[\mu',\mu'',\varphi']^* = T^{d'd''}_{-s'_x -s''_x}[-\mu',-\mu'',\varphi'+\pi], \quad (150)$$

where the indices d' and d'' are l or r, $\bar{l} = r$, $\bar{r} = l$, and the dependence of the amplitudes on the kinematic variables is explicitly shown. Note that Eqs. (149) and (150) are the consequences of the P-invariance and the T-invariance of Lagrangian (77), respectively.

We now have all the necessary ingredients to obtain the expression for the differential probability of the decay per unit time:

$$\frac{d\Gamma}{dp'_x dp''_x d\varphi'} = \frac{1}{(2\pi)^3}\frac{w\epsilon'\epsilon''}{\sqrt{3}}|\mathfrak{T}|^2, \quad (151)$$

where

$$|\mathfrak{T}|^2 = \sum_{s'_x,s''_x}|T^{ll}_{s'_x s''_x}|^2 = \sum_{s'_x,s''_x}|T^{rr}_{s'_x s''_x}|^2 \quad (152)$$

for $p'_x p''_x > 0$, and

$$|\mathfrak{T}|^2 = \sum_{s'_x,s''_x}|T^{lr}_{s'_x s''_x}|^2 = \sum_{s'_x,s''_x}|T^{rl}_{s'_x s''_x}|^2 \quad (153)$$

for $p'_x p''_x < 0$. Using Eq. (151), we obtain the angular distribution of the final fermion f_{p}:

$$\frac{d\Gamma}{d\theta' d\varphi'} = \int_{-p''_{x\,\max}}^{p''_{x\,\max}} \frac{d\Gamma}{dp'_x dp''_x d\varphi}\left|\frac{dp'_x}{d\theta'}\right| dp''_x, \quad (154)$$

where $p''_{x\,\max} = w^{-1}(3-2\sqrt{3}\nu)^{\frac{1}{2}}$, and the kinematic factor $dp'_x/d\theta'$ can be written as

$$\frac{dp'_x}{d\theta'} = -\frac{|\mathbf{p}'_{\parallel}|}{\sin^2(\theta') + 3^{-1/2}w\epsilon''\cos^2(\theta')}. \quad (155)$$

Of course, the angular distribution of the final antifermion $\bar{f}_{\mathrm{p}}$ coincides with that of the final fermion f_{p} because of the C-invariance of Lagrangian (77). From Eqs. (139)–(155), it follows that $d\Gamma/dp'_x dp''_x d\varphi'$ is invariant under the changes $p'_x \to -p'_x$, $p''_x \to -p''_x$, and that $d\Gamma/d\theta' d\varphi'$ is invariant under the change $\theta' \to \pi - \theta'$. We see that as in the previous cases, the angular distribution of the final fermion f_{p} (antifermion $\bar{f}_{\mathrm{p}}$) is invariant under the reflection about the domain wall's plane.

Figure 5 shows the dependence of the dimensionless combination $wg^{-2}d\Gamma/d\theta' d\varphi'$ on the fermion polar angle θ'. We see that the curves in Fig. 5 are similar to those in Fig. 3.

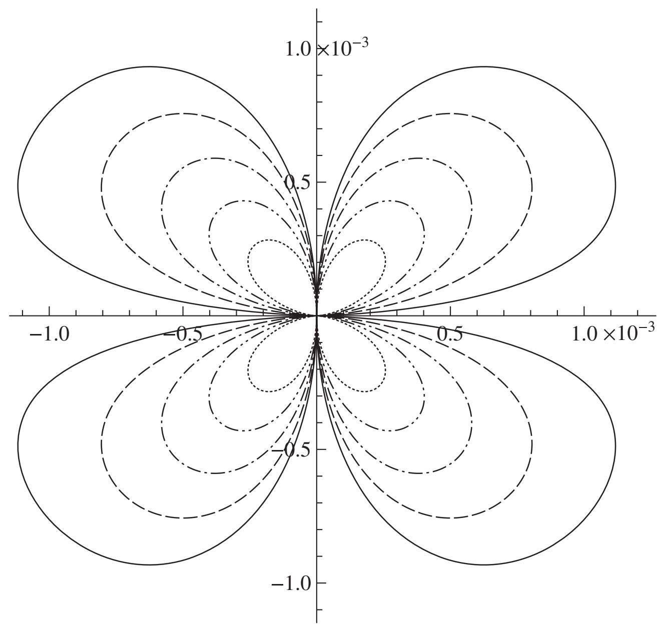

FIG. 5. Dependence of the dimensionless combination $wg^{-2}d\Gamma/d\theta' d\varphi'$ on the polar angle θ' of the final fermion f_{p}. The dependence is presented in polar coordinates $\rho = wg^{-2}d\Gamma/d\theta' d\varphi'$ and θ'. The solid, dashed, dash-dotted, dash-dot-dotted, and dotted curves correspond to the parameter $\nu = 0.1$, 0.2, 0.3, 0.4, and 0.5, respectively.

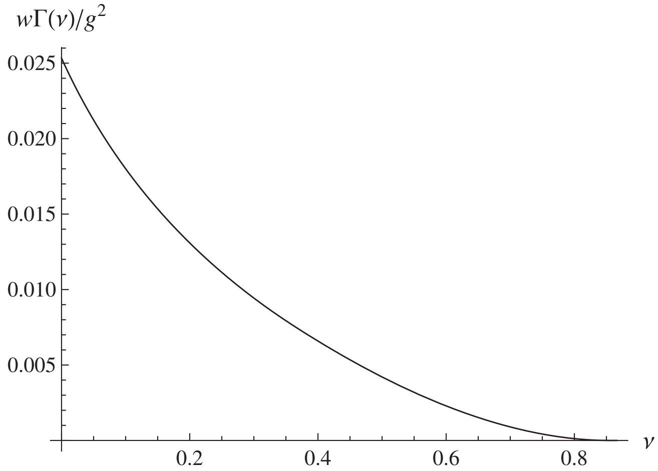

FIG. 6. Dependence of the dimensionless combination $w\Gamma/g^2$ on the parameter ν for the decay $\chi_1 \to f_{\mathrm{p}} + \bar{f}_{\mathrm{p}}$.

The vanishing of $d\Gamma/d\theta' d\varphi'$ at $\theta' = \pi/2$ (i.e., at $\mu' = 0$) is due to the vanishing of the transition coefficient $\mathcal{T}(\nu, \mu')$ at $\mu' = 0$. The reason for the vanishing of $d\Gamma/d\theta' d\varphi'$ at $\theta' = 0,\ \pi$ (i.e., at $|\mathbf{p}'_{\parallel}| = 0$) is that the kinematic factor $dp'_x/d\theta'$ vanishes at $|\mathbf{p}'_{\parallel}| = 0$.

The decay width $\Gamma = \int (d\Gamma/d\theta' d\varphi') d\theta' d\varphi'$ can be obtained from Eq. (154) by numerical methods. Figure 6 presents the dependence of the dimensionless combination $w\Gamma/g^2$ on the parameter ν in the kinematically allowable domain $(0, \sqrt{3}/2)$. We see that the decay width $\Gamma(\nu)$ vanishes as ν tends to the right boundary point $\sqrt{3}/2$. But unlike the decay width $\Gamma(\nu)$ in Fig. 4, the decay width $\Gamma(\nu)$ in Fig. 6 does not vanish as ν tends to zero, because the massless mode's suppression is absent in this case. It was found numerically that

$$wg^{-2}\Gamma(\nu) \approx 0.025 - 0.09\nu \quad \text{as} \quad \nu \to 0, \tag{156}$$

and

$$wg^{-2}\Gamma(\nu) \approx 1.335\left(\nu - \frac{\sqrt{3}}{2}\right)^3 \quad \text{as} \quad \nu \to \frac{\sqrt{3}}{2}. \tag{157}$$

V. CONCLUSIONS

In the present paper, the decays of excited bosonic and fermionic modes on the domain wall have been investigated. Certain analytical and numerical results were obtained. In particular, the analytical expressions of the wave functions of the excited localized fermionic modes were obtained, as well as those of the nonlocalized fermionic modes. The analytical expressions of the reflection and transmission coefficients were obtained for fermion scattering from the domain wall. For certain decay channels, analytical expressions of the amplitudes, angular distributions, and decay widths were found. The widths of the decays $f_1 \to f_0 + \chi_{\mathrm{k}}, \chi_1 \to \bar{f}_0 + f_{\mathrm{p}}$, and $\chi_1 \to \bar{f}_{\mathrm{p}} + f_{\mathrm{p}}$ were obtained numerically.

To obtain these results, a number of approximations were used. The bosonic wave functions of Sec. III A were obtained in the weak coupling approximation $\lambda \ll 1$. The fermionic wave functions of Sec. III B were obtained within the external field approximation in which we neglect the backreaction of the fermions on the domain wall. The condition for the validity of this approximation is the smallness of the Yukawa coupling constant: $g \ll 1$. Finally, the decay amplitudes were calculated at the first order in the coupling constant g and λ; this fact also requires the fulfillment of the conditions $g \ll 1$ and $\lambda \ll 1$. Note that as $\lambda \ll 1$ and $g \ll 1$, the dimensionless combination $\nu = gw\eta = \sqrt{2}g/\sqrt{\lambda}$ can, in principle, have an arbitrary value.

Two-body scattering of mesons and fermions on the domain wall can also be studied, as well as fermion-antifermion annihilation. This requires the analytical expressions for the mesonic and fermionic propagators in the external field of the domain wall. The analytical expression can be obtained for the mesonic propagator, while we were unable to obtain the analytical expression for the fermionic propagator.

The domain wall (antiwall) is the $(3+1)$-dimensional analog of the $(1+1)$-dimensional kink (antikink). It is well known [13,15] that the kink possesses bosonic and fermionic zero modes. The massless bosonic and massless fermionic modes living on the domain wall are the $(3+1)$-dimensional analogs of the kink's zero modes.

The bosonic zero mode of the $(1+1)$-dimensional kink is an isolated normalizable eigenfunction with a zero eigenvalue. Unlike other bosonic modes of the kink that are vibrational modes, the bosonic zero mode is a translational mode. This is because the kink breaks the translational symmetry of the model's Lagrangian. The excitation of this mode does not lead to an increase of the kink's mass, but rather to a relativistic increase of the kink's kinetic energy [34,44]. The presence of the isolated normalizable zero mode in the spectrum of bosonic fluctuations leads to technical difficulties in the calculation of high-order quantum corrections to the kink's mass [44,45]. Unlike the kink, the domain wall has no normalized bosonic modes that are isolated in the functional space. Instead, the massless bosonic modes living on the domain wall are the family of the eigenfunctions that are continuously parametrized by the two-dimensional momentum $\mathbf{k}_{\parallel}$. These modes are not normalized in the ordinary sense, but instead are normalized to $(2\pi)^2(2\omega)^{-1}\delta^{(2)}(\mathbf{k}_{\parallel} - \mathbf{k}'_{\parallel})$. Unlike the kink's zero mode, the domain wall's massless modes are vibrational; i.e., the excitation of these modes leads to an increase of the energy in the rest frame of the domain wall. The domain wall's massless modes become the zero mode in the limit $\mathbf{k}_{\parallel} \to \mathbf{0}$. Note that the domain wall's zero mode is the lower limit of the continuum of eigenvalues, while the kink's zero mode is the term of the discrete spectrum of eigenvalues. Unlike the kink's zero mode, the domain wall's zero mode cannot be excited physically, because the $(3+1)$-dimensional domain wall has infinite mass.

It was shown in Refs. [13,15] that the $(1+1)$-dimensional kink has exactly one fermionic zero mode that can be normalized to unity. This mode is invariant under the charge conjugation. These properties of the kink's fermionic zero mode lead to the fractionalization of the fermionic charge of the kink-fermion system [15]. In contrast to the kink, the domain wall has no normalized fermionic modes that are isolated in the functional space from other fermionic modes. Instead, there are two families (i.e., the massless fermionic and massless antifermionic modes) that are continuously parametrized by the two-dimensional momentum $\mathbf{p}_{\parallel}$. These modes are not normalized to unity, but instead are normalized to $(2\pi)^2\delta^2(\mathbf{p}_{\parallel}-\mathbf{p}'_{\parallel})$. Of course, the massless fermionic and massless antifermionic modes are not invariant under the charge conjugation, but instead the two families of the massless modes are related to each other by charge conjugation (62). These massless modes become zero modes [Eq. (68)] in the limit $\mathbf{p}_{\parallel}\to\mathbf{0}$. The two linear combinations $2^{-1/2}(\Psi_0^{+}+\Psi_0^{-})$ and $2^{-1/2}(\Psi_0^{+}-\Psi_0^{-})$ can be formed from these zero modes. One of them is invariant under the charge conjugation, while the other changes the sign. However, the presence of the zero modes does not lead to a degeneracy of the ground state of the domain wall. This is because these zero modes are part of the continuous spectrum of the Dirac Hamiltonian. Therefore, the fermion cannot have an energy that is exactly equal to zero. Instead, the fermion's energy is in the range $(0,\epsilon)$, where ϵ can be arbitrarily small, but not equal to zero. Hence, interaction with the fermions does not lead to degeneracy of the domain wall's ground state.

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