AN EFFICIENT MODIFIED MULTI-STEP METHOD FOR SOLVING BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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ЭФФЕКТИВНЫЙ МОДИФИЦИРОВАННЫЙ МНОГОШАГОВЫЙ МЕТОД ДЛЯ РЕШЕНИЯ ГРАНИЧНЫХ ЗАДАЧ ДЛЯ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Аннотация. *В статье предложен модифицированный многошаговый метод для прямого решения краевых задач для обыкновенных дифференциальных уравнений высших порядков. Дискретная форма предлагаемого метода была получена с использованием коллокационного подхода. Для вычисления непрерывных коэффициентов метода использованы три сеточные и одна внесеточная точка. Численные эксперименты продемонстрировали эффективность предложенного метода по сравнению с результатами, известными из литературы.*

Introduction. The application of mathematical formulation in science and engineering is given by boundary value problem (BVP):

$$y''' = f(x, y, y', y''), \qquad y(a) = y_0, \qquad y'(a) = \delta_0, \qquad y(b) = y_M. \tag{1}$$

$$y''' = f(x, y, y', y''), \qquad y(a) = y_0, \qquad y'(a) = \delta_0, \qquad y'(b) = y_M. \tag{2}$$

There are many methods to solve (1), (2). Most of these methods are solved BVP by reducing a higher order ordinary differential equations (ODEs) to an equivalent system of first order ODEs which take a lot of time and human effort. Alternative approach is to solve higher ODEs directly. In the paper [1] authors investigated two- and three-stage Runge-Kutta type methods for special third order ODEs. Higher order linear multi-step methods were proposed by Jator [2] to the numerical integration of third order BVP. The case of the four-points block hybrid collocation method with two- and three-step points to solve general third order ODEs directly was studied by Yap and Ismail [3].

In this research, we develop a continuous hybrid linear multi-step method (HLMM) for direct solution of BVPs without reducing the problem to a lower order system or to an initial value problem (IVP) equivalent. The proposed HLMM is zero stable, consistent and more accurate than the existing one. Experimental results confirm the superiority of the new schemes over the existing methods.

Construction Method. Our objective is to derive hybrid linear multi-step method (HLMM) of the form

$$\sum_{j=0}^{r-1} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j y_{n+j} + h^3 \beta_\eta f_{n+\eta} \tag{3}$$

where α_j, β_j, β_η are unknown constants and $\eta \in \mathbf{R}$. We note that $\alpha_k = 1$, $\beta_k \neq 0$, and α_0 and β_0 are not zero at the same time. In order to obtain form (3), we proceed seeking to approximate the exact solution $y(x)$ of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{4}$$

where a_j are parameters to be determined, $r = k$ and $s > 0$ are the number of interpolation and collocation points respectively. Then we construct our continuous approximation by imposing the following conditions

$$y(x_{n+j}) = y_{n+j}, \qquad y'''(x_{n+j}) = f_{n+j}, \qquad j = 0, 1, 2, \ldots, r-1 \tag{5}$$

Equation (5) result in $(r+s)$-systems of equations which can be solved with the inverse matrix method to obtain the value of a_j. Our continuous approximation is constructed by substituting the values a_j into equation (4). The method of continuous approximation can be expressed as

$$y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^3 \sum_{j=0}^{k} \beta_j(x) y_{n+j} + h^3 \beta_\eta(x) f_{n+\eta} \tag{6}$$

where $\alpha_j(x)$, $\beta_j(x)$, $\beta_\eta(x)$ are continuous coefficients.

Three-Step Hybrid Method with One Off-step Collocation Point. We use Equation (6) to obtain a three-step HLMM with the following specification: $r = 3$, $s = 5$, $\eta = 8/3$, $k = 3$, $\alpha_j(x)$, $\beta_j(x)$, $\beta_\eta(x)$ can be expressed as functions of $t = (x - x_n)/h$. The HLMMs are usually represented in the form of a single block r-point multi-step method of the form [4]

$$\boldsymbol{A}^{(0)}\mathbf{Y}_m = \boldsymbol{A}^{(1)}\boldsymbol{Y}_{m-1} + h^3 \boldsymbol{B}^{(0)}\boldsymbol{F}_m + h^3 \boldsymbol{B}^{(1)}\boldsymbol{F}_{m-1} \tag{7}$$

where h is a fixed mesh size within a block hybrid method, $\boldsymbol{A}^{(0)}$ is an identity matrix, $\mathbf{Y}_m$, $\boldsymbol{Y}_{m-1}$, $\boldsymbol{F}_m$, $\boldsymbol{F}_{m-1}$ are arrays (vectors) of numerical approximation.

The hybrid method can be significantly shown in the form of equation (7):

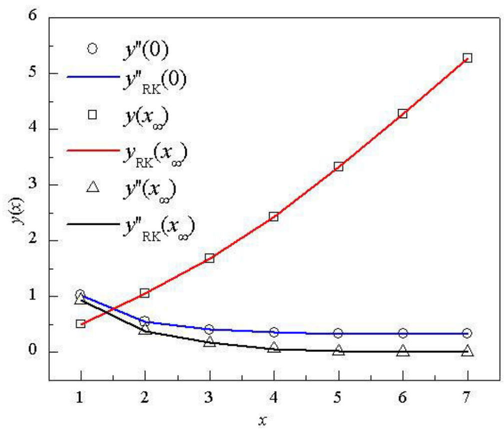

Fig.1. Comparison of numerical solutions of proposed method with Runge-Kutta method for the Problem 1

$$\begin{pmatrix} 3 & -3 & 0 & 1 \\ \frac{16}{9} & -\frac{20}{9} & 1 & 0 \\ -2 & \frac{1}{2} & 0 & 0 \\ 2 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{8}{3}} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{5}{9} \\ 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

$$h^3 \begin{pmatrix} \frac{47}{100} & \frac{23}{40} & -\frac{81}{800} & \frac{1}{20} \\ \frac{7}{27} & \frac{203}{729} & -\frac{25}{324} & -\frac{65}{2187} \\ \frac{1399}{4200} & -\frac{23}{168} & -\frac{783}{5600} & -\frac{17}{2800} \\ -\frac{109}{120} & \frac{61}{120} & -\frac{81}{160} & \frac{79}{360} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{8}{3}} \\ f_{n+3} \end{pmatrix} + h^3 \begin{pmatrix} 0 & 0 & 0 & \frac{1}{160} \\ 0 & 0 & 0 & \frac{31}{8748} \\ 0 & 0 & 0 & -\frac{13}{224} \\ 0 & 0 & 0 & -\frac{451}{1440} \end{pmatrix} \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix}. \tag{8}$$

Numerical Experiments. The accuracy of the proposed method was implemented for direct solution of BVPs of third order ODEs of linear and non-linear equations. The implementation of the method was coded using Maple Software.

Table 1

Numerical Results for Problem 2

x	Exact Solution	Numerical Solution	Error	Error [6]
0.0	-0.01210709	-0.012107056410	3.36×10^{-8}	6.6530×10^{-5}
0.1	-0.01126851	-0.011268451800	5.82×10^{-8}	6.5000×10^{-5}
0.2	-0.00922221	-0.009222146339	6.37×10^{-8}	5.2254×10^{-5}
0.3	-0.00646687	-0.006466811798	5.82×10^{-8}	3.6300×10^{-5}
0.4	-0.00332019	-0.003320153971	3.60×10^{-8}	1.8750×10^{-5}
0.6	0.00332019	0.003320118521	7.15×10^{-8}	1.7340×10^{-5}
0.7	0.00646687	0.006466679335	1.91×10^{-7}	3.4050×10^{-5}
0.8	0.00922221	0.009221906116	3.04×10^{-7}	4.9801×10^{-5}
0.9	0.01126851	0.011268057530	4.52×10^{-7}	6.2020×10^{-5}
1.0	0.01210709	0.012106500950	5.89 10 7	6.3480 10 5

Problems 1. Non-linear Blasius Equation [5]: $2y'''+yy''=0$, $y(0)=0$, $y'(0)=0$, $y'(\infty)=1$.

The exact solution does not exist. Comparison of numerical solutions using the proposed method and the fourth order Runge-Kutta (RK) method for the Problem 1 is shown in Fig. 1.

Problems 2. Sandwich Beam Problem [6]: $y'''+k^2y'+r=0$, $y(0.5)=0$, $y'(0)=0$, $y'(1)=0$.

The proposed schemes for the values k=1 and r=5 are relatively more accurate than the schemes of Tirmizi et al. (2005) [6] for Problem 2. Absolute errors are presented in Table 1.

Conclusion. This research describes the development, analysis and implementation of block methods for solving third order ordinary differential equations directly. The development and/or construction of class of hybrid linear multi-step methods for direct solution of initial value problems and boundary value problems arising from third order ODEs have been presented. The derived schemes which are of block form were analyzed and applied to some selected and standard problems from literature.

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