

One-dimensional soliton system of gauged Q -ball and anti- Q -ball

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The $(1 + 1)$ -dimensional gauge model of two complex self-interacting scalar fields that interact with each other through an Abelian gauge field and a quartic scalar interaction is considered. It is shown that the model has nontopological soliton solutions describing soliton systems consisting of two Q -ball components possessing opposite electric charges. The two Q -ball components interact with each other through the Abelian gauge field and the quartic scalar interaction. The interplay between the attractive electromagnetic interaction and the repulsive quartic interaction leads to the existence of symmetric and nonsymmetric soliton systems. Properties of these systems are investigated by analytical and numerical methods. The symmetric soliton system exists in the whole allowable interval of the phase frequency, whereas the nonsymmetric soliton system exists only in some interior subinterval. Despite the fact that these soliton systems are electrically neutral, they nevertheless possess nonzero electric fields in their interiors. It is found that the nonsymmetric soliton system is more preferable from the viewpoint of energy than the symmetric one. Both symmetric and nonsymmetric soliton systems are stable against decay into massive scalar bosons.

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I. INTRODUCTION

There are many field models possessing global symmetries and corresponding conserved Noether charges that admit the existence of nontopological solitons [1,2]. The determining property of a nontopological soliton is that it is an extremum of the energy functional at a fixed value of the Noether charge. This feature of nontopological solitons leads to the characteristic time dependence $\propto \exp(-i\omega t)$ of their fields. This nontrivial time dependence of the soliton's field allows one to avoid severe restrictions of Derrick's theorem [3], so scalar nontopological solitons can exist in any number of spatial dimensions.

The simplest nontopological soliton, proposed in Ref. [4] and known as a Q -ball [5], has been found in a model of a complex scalar field possessing a global $U(1)$ symmetry. Q -balls can also exist in scalar field models possessing a global non-Abelian symmetry [6,7]. They are present [8,9] in supersymmetric extensions of the Standard Model having flat directions in the interaction potential of scalar fields. Such Q -balls are of great interest to cosmological models describing the evolution of the early Universe [10–15].

There are also other types of nontopological solitons in global-symmetric field models. The most known of them is the nontopological soliton of the Friedberg-Lee-Sirlin model [16]. The model consists of two interacting scalar fields, one of which is real and the other of which is complex. It possesses a global $U(1)$ symmetry and a renormalizable interaction potential. Another example is the nontopological soliton in the model of a massive self-interacting complex vector field [17].

In all of the examples given above, the existence of nontopological solitons is due to a global invariance of the corresponding Lagrangians, so the Noether charge of such solitons cannot be a source of a gauge field. At the same time, nontopological solitons also exist in field models possessing a local gauge invariance, both Abelian [18–26] and non-Abelian [27–29]. The nontopological solitons [18–26,29] possess a long-range gauge electric field, so Noether charges of these solitons are proportional to their electric charges. In particular, electrically charged Q -balls [25,26] exist in models with a scalar self-interaction potential resulting from gauge-mediated supersymmetry breaking and may play an important role in cosmology. However, all these electrically charged nontopological solitons are three-dimensional ones. This is because any one-dimensional or two-dimensional field configuration with a nonzero electric charge possesses infinite energy, as follows from Gauss's law and the expression for the electric field energy density. Nevertheless, there are electrically neutral low-dimensional soliton systems that have a nonzero electric field in their interiors. In particular, the

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two-dimensional soliton systems consisting of vortex and Q -ball components interacting through an Abelian gauge field have been described in Refs. [30,31].

In the present paper, we research the $(1+1)$ -dimensional gauge model of two complex self-interacting scalar fields interacting with each other through an Abelian gauge field and a quartic scalar interaction. In particular, it is found that symmetric and nonsymmetric soliton systems exist in the model. The soliton systems consist of two Q -ball components having opposite electric charges. The soliton systems are electrically neutral but nevertheless possess nonzero electric fields in their interiors. The paper is structured as follows. In Sec. II, we describe briefly the Lagrangian and the field equations of the model under consideration. By means of the Hamiltonian formalism and the Lagrange multipliers method, the time dependence is established for the soliton system's fields. Then, we give the ansatz used for solving the model's field equations and establish the basic relation for the nontopological soliton system. In Sec. III, we derive the system of nonlinear differential equations for the ansatz functions and the expressions for the electromagnetic current density and the energy density in terms of these functions. Then, some general properties of the soliton system are established, its asymptotic properties are researched, and the virial relation for the soliton system is derived. In Sec. IV, we study properties of the soliton system in the thick-wall and thin-wall regimes and establish its stability against decay into free massive scalar bosons. In Sec. V, we briefly describe the procedure for numerical solving of a boundary value problem and discuss possible types of soliton solutions of the problem. The dependences of the energy and the Noether charge on the phase frequency are presented for both (symmetric and nonsymmetric) types of the soliton solutions. Then, we show the dependences of the symmetric soliton system's energy and the energy difference between the symmetric and nonsymmetric soliton systems on the Noether charge. After that, we present the numerical results for the ansatz functions, the energy density, the electric charge density, and the electric field strength for the symmetric and nonsymmetric soliton systems.

Throughout the paper, the natural units $c = 1$, $\hbar = 1$ are used.

II. LAGRANGIAN AND THE FIELD EQUATIONS

The $(1+1)$ -dimensional gauge model we are interested in is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*D^\mu\phi + (D_\mu\chi)^*D^\mu\chi - V(|\phi|) - U(|\chi|) - W(|\phi|, |\chi|). \quad (1)$$

It describes the two complex scalar fields ϕ and χ that minimally interact with the Abelian gauge field A_μ through the covariant derivatives:

$$D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi, \quad D_\mu\chi = \partial_\mu\chi + iqA_\mu\chi. \quad (2)$$

The scalar fields interact with each other and self-interact. The self-interaction potentials of the scalar fields have the form

$$V(|\phi|) = m_\phi^2|\phi|^2 - \frac{g_\phi}{2}|\phi|^4 + \frac{h_\phi}{3}|\phi|^6, \quad (3)$$

$$U(|\chi|) = m_\chi^2|\chi|^2 - \frac{g_\chi}{2}|\chi|^4 + \frac{h_\chi}{3}|\chi|^6, \quad (4)$$

whereas the interaction potential is

$$W(|\phi|, |\chi|) = \lambda|\phi|^2|\chi|^2. \quad (5)$$

We suppose that the self-interaction potentials V and U admit the existence of usual nongauged nontopological solitons (Q -balls) formed from the scalar fields ϕ and χ , respectively. We also suppose that the potentials V and U possess global minima at $\phi = 0$ and $\chi = 0$, respectively. Then, the parameters of the potentials satisfy the condition

$$\frac{m_i^2 h_i}{g_i^2} > \frac{3}{16}, \quad (6)$$

where the index $i = (\phi, \chi)$.

The Lagrangian (1) is invariant under the local gauge transformations:

$$\phi(x, t) \rightarrow \phi'(x, t) = \exp(-ie\Lambda(x, t))\phi(x, t), \quad (7a)$$

$$\chi(x, t) \rightarrow \chi'(x, t) = \exp(-iq\Lambda(x, t))\chi(x, t), \quad (7b)$$

$$A_\mu(x, t) \rightarrow A'_\mu(x, t) = A_\mu(x, t) + \partial_\mu\Lambda(x, t). \quad (7c)$$

At the same time, it is also invariant under the two independent global gauge transformations:

$$\phi(x, t) \rightarrow \phi'(x, t) = \exp(-i\alpha)\phi(x, t), \quad (8a)$$

$$\chi(x, t) \rightarrow \chi'(x, t) = \exp(-i\beta)\chi(x, t). \quad (8b)$$

The Noether currents corresponding to transformations (8) are written as

$$j_\phi^\mu = i[\phi^*D^\mu\phi - (D^\mu\phi)^*\phi], \quad (9a)$$

$$j_\chi^\mu = i[\chi^*D^\mu\chi - (D^\mu\chi)^*\chi]. \quad (9b)$$

The presence of the two separately conserved Noether charges $Q_\phi = \int_{-\infty}^{\infty} j_\phi^0 dx$ and $Q_\chi = \int_{-\infty}^{\infty} j_\chi^0 dx$ is the result of the structure of the interaction potential W and the neutrality of the Abelian gauge field A_μ .

The field equations of the model are obtained by varying the action $S = \int \int_{-\infty}^{\infty} \mathcal{L} dx dt$ in the corresponding fields,

$$D_{\mu} D^{\mu} \phi + \frac{\partial V}{\partial |\phi|} \frac{\phi}{2|\phi|} + \frac{\partial W}{\partial |\phi|} \frac{\phi}{2|\phi|} = 0, \quad (10)$$

$$D_{\mu} D^{\mu} \chi + \frac{\partial U}{\partial |\chi|} \frac{\chi}{2|\chi|} + \frac{\partial W}{\partial |\chi|} \frac{\chi}{2|\chi|} = 0, \quad (11)$$

$$\partial_{\mu} F^{\mu\nu} = j^{\nu}, \quad (12)$$

where the electromagnet current j^{ν} is written in terms of two Noether currents (9),

$$j^{\nu} = e j_{\phi}^{\nu} + q j_{\chi}^{\nu}. \quad (13)$$

The symmetric energy-momentum tensor of the model is written as

$$\begin{aligned} T_{\mu\nu} = & -F_{\mu\lambda} F_{\nu}{}^{\lambda} + \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \\ & + (D_{\mu}\phi)^* D_{\nu}\phi + (D_{\nu}\phi)^* D_{\mu}\phi \\ & + (D_{\mu}\chi)^* D_{\nu}\chi + (D_{\nu}\chi)^* D_{\mu}\chi \\ & - g_{\mu\nu} [(D_{\mu}\phi)^* D^{\mu}\phi + (D_{\mu}\chi)^* D^{\mu}\chi \\ & - V(|\phi|) - U(|\chi|) - W(|\phi|, |\chi|)], \end{aligned} \quad (14)$$

so we have the expression for the energy density

$$\begin{aligned} T_{00} = \mathcal{E} = & \frac{1}{2} E_x^2 + (D_t\phi)^* D_t\phi + (D_x\phi)^* D_x\phi \\ & + (D_t\chi)^* D_t\chi + (D_x\chi)^* D_x\chi \\ & + V(|\phi|) + U(|\chi|) + W(|\phi|, |\chi|), \end{aligned} \quad (15)$$

where $E_x = F_{01} = \partial_t A_1 - \partial_x A_0$ is the electric field strength.

Field equation (12) with $\nu = 0$ is Gauss's law. It can be written in the form

$$\partial_x E_x = j_0. \quad (16)$$

Integrating Eq. (16) over $x \in (-\infty, \infty)$, we obtain the relation

$$\lim_{x \rightarrow \infty} E_x(x, t) - \lim_{x \rightarrow -\infty} E_x(x, t) = Q, \quad (17)$$

where $Q = \int_{-\infty}^{\infty} j^0 dx$ is the electric charge of a field configuration. Further, from Eq. (15), it follows that the electric field's energy is

$$E^{(E)} = \frac{1}{2} \int_{-\infty}^{\infty} E_x^2 dx. \quad (18)$$

We see from Eq. (18) that the necessary condition for the finiteness of $E^{(E)}$ is $\lim_{x \rightarrow \pm\infty} E_x(x, t) = 0$. Then, from

Eq. (17), it follows that the electric charge of a finite energy field configuration vanishes. Thus, we conclude that in (1 + 1) dimensions any field configuration possessing finite energy must have zero electric charge. Note that in (2 + 1) dimensions any field configuration with finite energy also has zero electric charge unless the Chern-Simons term is present in the Lagrangian. In (3 + 1) dimensions, however, there are electrically charged objects with finite energy (e.g., electrically charged Q -ball or dyon).

By analogy with nontopological solitons, we want to find a solution of model (1) that is an extremum of the energy functional $E = \int_{-\infty}^{\infty} \mathcal{E} dx$ at fixed values of the Noether charges $Q_{\phi} = \int_{-\infty}^{\infty} j_{\phi}^0 dx$ and $Q_{\chi} = \int_{-\infty}^{\infty} j_{\chi}^0 dx$. Such a solution is an unconditional extremum of the functional

$$\begin{aligned} F = & \int_{-\infty}^{\infty} \mathcal{E} dx - \omega_{\phi} \int_{-\infty}^{\infty} j_{\phi}^0 dx - \omega_{\chi} \int_{-\infty}^{\infty} j_{\chi}^0 dx \\ = & E - \omega_{\phi} Q_{\phi} - \omega_{\chi} Q_{\chi}, \end{aligned} \quad (19)$$

where ω_{ϕ} and ω_{χ} are the Lagrange multipliers. However, a finite energy field configuration must be electrically neutral, so we have the relation

$$Q = e Q_{\phi} + q Q_{\chi} = 0. \quad (20)$$

We see that the Noether charges Q_{ϕ} and Q_{χ} are not independent, so Eq. (19) can be rewritten in the two equivalent forms

$$F = E - \omega Q_{\chi} = E - \tilde{\omega} Q_{\phi}, \quad (21)$$

where $\omega = \omega_{\chi} - q e^{-1} \omega_{\phi}$ and $\tilde{\omega} = \omega_{\phi} - e q^{-1} \omega_{\chi}$ are new Lagrange multipliers. Thus, the functional F can be expressed in terms of either Q_{χ} or Q_{ϕ} . In the present paper, we shall use the first variant: $F = E - \omega Q_{\chi}$.

To determine the time dependence of the soliton solution, we will use the Hamiltonian formalism. We adopt the axial gauge in which the spatial component of the gauge potential vanishes: $A_x = A^1 = 0$. In this case, the gauge model is described in terms of the eight canonically conjugated fields: ϕ , $\pi_{\phi} = (D_0\phi)^*$, ϕ^* , $\pi_{\phi^*} = D_0\phi$, χ , $\pi_{\chi} = (D_0\chi)^*$, χ^* , and $\pi_{\chi^*} = D_0\chi$. Then, the Hamiltonian density has the form

$$\begin{aligned} \mathcal{H} = & \pi_{\phi} \partial_t \phi + \pi_{\phi^*} \partial_t \phi^* + \pi_{\chi} \partial_t \chi + \pi_{\chi^*} \partial_t \chi^* - \mathcal{L} \\ = & -\frac{1}{2} (\partial_x A_0)^2 + \pi_{\phi} \pi_{\phi^*} + \pi_{\chi} \pi_{\chi^*} + \partial_x \phi^* \partial_x \phi + \partial_x \chi^* \partial_x \chi \\ & + i e A_0 \{ \phi^* \pi_{\phi^*} - \phi \pi_{\phi} \} + i q A_0 \{ \chi^* \pi_{\chi^*} - \chi \pi_{\chi} \} \\ & + V(|\phi|) + U(|\chi|) + W(|\phi|, |\chi|), \end{aligned} \quad (22)$$

where the time component A_0 is determined in terms of the canonically conjugated fields by Gauss's law,

$$\partial_x^2 A_0 + ie\{\phi^* \pi_{\phi^*} - \phi \pi_{\phi}\} + iq\{\chi^* \pi_{\chi^*} - \chi \pi_{\chi}\} = 0. \quad (23)$$

Note that energy density (15) does not coincide with Hamiltonian density (22):

$$\begin{aligned} \mathcal{H} - \mathcal{E} = & -(\partial_x A_0)^2 + ieA_0\{\phi^* \pi_{\phi^*} - \phi \pi_{\phi}\} \\ & + iqA_0\{\chi^* \pi_{\chi^*} - \chi \pi_{\chi}\}. \end{aligned} \quad (24)$$

However, the integral of Eq. (24) over the space dimension vanishes for field configurations possessing finite energy and satisfying Gauss's law (23). So, for such configurations,

$$E = \int_{-\infty}^{\infty} \mathcal{E} dx = H = \int_{-\infty}^{\infty} \mathcal{H} dx. \quad (25)$$

It can be shown that in the adopted gauge the field equations (10) and (11) can be rewritten in the Hamiltonian form:

$$\partial_t \phi = \frac{\delta H}{\delta \pi_{\phi}} = \frac{\delta E}{\delta \pi_{\phi}}, \quad \partial_t \pi_{\phi} = -\frac{\delta H}{\delta \phi} = -\frac{\delta E}{\delta \phi}, \quad (26)$$

$$\partial_t \chi = \frac{\delta H}{\delta \pi_{\chi}} = \frac{\delta E}{\delta \pi_{\chi}}, \quad \partial_t \pi_{\chi} = -\frac{\delta H}{\delta \chi} = -\frac{\delta E}{\delta \chi}. \quad (27)$$

Further, the first variation of the functional F vanishes in a neighborhood of the soliton solution,

$$\delta F = \delta E - \omega \delta Q_{\chi} = 0, \quad (28)$$

where the first variation of the Noether charge Q_{χ} can be expressed in terms of the canonically conjugated fields:

$$\delta Q_{\chi} = -i \int_{-\infty}^{\infty} (\pi_{\chi} \delta \chi + \chi \delta \pi_{\chi} - \text{c.c.}) dx. \quad (29)$$

From Eqs. (26)–(29), we obtain the Hamilton field equations

$$\partial_t \chi = \frac{\delta E}{\delta \pi_{\chi}} = \omega \frac{\delta Q_{\chi}}{\delta \pi_{\chi}} = -i\omega \chi, \quad (30)$$

$$\partial_t \chi^* = \frac{\delta E}{\delta \pi_{\chi^*}} = \omega \frac{\delta Q_{\chi}}{\delta \pi_{\chi^*}} = i\omega \chi, \quad (31)$$

while time derivatives of the other model's fields are equal to zero. Thus, in the adopted gauge $A_x = 0$, only the scalar field χ has the nontrivial time dependence, whereas the model's fields ϕ and A_0 do not depend on time:

$$\phi(x, t) = f(x), \quad (32a)$$

$$\chi(x, t) = s(x) \exp(-i\omega t), \quad (32b)$$

$$A_{\mu}(x, t) = (a_0(x), 0). \quad (32c)$$

Note that, instead of the expression $F = E - \omega Q_{\chi}$, we may use the equivalent one, $F = E - \tilde{\omega} Q_{\phi}$. It can be shown that in this case the time dependences of the fields are obtained from those of Eqs. (32) by local gauge transformation (7) with the parameter $\Lambda = -q^{-1}\omega t$.

From extremum condition (28), it follows that the soliton solution satisfies the important relation

$$\frac{dE}{dQ_{\chi}} = \omega, \quad (33)$$

where the Lagrange multiplier ω is understood as some function of the Noether charge Q_{χ} . Note that, unlike Eqs. (32), relation (33) is gauge invariant. Just as in the case of nongauged nontopological solitons [1], Eq. (33) plays the primary role in the determining of properties of the gauged nontopological soliton system.

III. SOME PROPERTIES OF THE SOLUTION

In Eqs. (32), $f(x)$ and $s(x)$ are some complex functions of the real argument x . Substituting Eqs. (32) into field equations (10)–(12), we obtain the system of ordinary nonlinear differential equations for the functions $a_0(x)$, $f(x)$, and $s(x)$:

$$a_0''(x) - 2(e^2|f(x)|^2 + q^2|s(x)|^2)a_0(x) + 2q\omega|s(x)|^2 = 0, \quad (34)$$

$$\begin{aligned} f''(x) - (m_{\phi}^2 - e^2 a_0(x)^2)f(x) \\ + (g_{\phi}|f(x)|^2 - h_{\phi}|f(x)|^4 - \lambda|s(x)|^2)f(x) = 0, \end{aligned} \quad (35)$$

$$\begin{aligned} s''(x) - (m_{\chi}^2 - (\omega - qa_0(x))^2)s(x) \\ + (g_{\chi}|s(x)|^2 - h_{\chi}|s(x)|^4 - \lambda|f(x)|^2)s(x) = 0. \end{aligned} \quad (36)$$

From Eq. (35), it follows that the real and imaginary parts of $f(x)$ satisfy the same differential equation, whereas Eq. (36) leads us to the same conclusion for the function $s(x)$. This in turn means that the functions $f(x)$ and $s(x)$ can be written as $f(x) = \exp(i\alpha)|f(x)|$ and $s(x) = \exp(i\beta)|s(x)|$, where α and β are real constant phases. These phases, however, can be gauged away by global gauge transformations (8). Thus, we can suppose without loss of generality that $f(x)$ and $s(x)$ are real functions of x .

Substituting Eqs. (32) into Eqs. (13) and (15), we obtain the electromagnetic current density and the energy density in terms of the real functions $a_0(x)$, $f(x)$, and $s(x)$:

$$j^{\mu} = (2q\omega s^2 - 2(q^2 s^2 + e^2 f^2)a_0, 0), \quad (37)$$

$$\begin{aligned} \mathcal{E} = & \frac{a_0'^2}{2} + f'^2 + s'^2 + (\omega - qa_0)^2 s^2 + e^2 a_0^2 f^2 \\ & + V(f) + U(s) + W(f, s). \end{aligned} \quad (38)$$

The finiteness of the soliton system's energy $E = \int_{-\infty}^{\infty} \mathcal{E} dx$ leads to the following boundary conditions for the functions $a_0(x)$, $f(x)$, and $s(x)$:

$$a'_0(x) \xrightarrow{x \rightarrow -\infty} 0, \quad a'_0(x) \xrightarrow{x \rightarrow \infty} 0, \quad (39a)$$

$$f(x) \xrightarrow{x \rightarrow -\infty} 0, \quad f(x) \xrightarrow{x \rightarrow \infty} 0, \quad (39b)$$

$$s(x) \xrightarrow{x \rightarrow -\infty} 0, \quad s(x) \xrightarrow{x \rightarrow \infty} 0. \quad (39c)$$

Let us discuss some general properties of the soliton system. The invariance of the Lagrangian (1) under the charge conjugation leads to the invariance of system (34)–(36) under the discrete transformation

$$\omega, a_0, f, s \rightarrow -\omega, -a_0, f, s. \quad (40)$$

From Eqs. (37), (38), and (40), it follows that the energy E is an even function of ω , whereas the Noether charges Q_ϕ and Q_χ are odd functions of ω :

$$E(-\omega) = E(\omega), \quad (41)$$

$$Q_{\phi,\chi}(-\omega) = -Q_{\phi,\chi}(\omega). \quad (42)$$

The Lagrangian (1) is also invariant under the parity transformation. It follows that system (34)–(36) is invariant under the space inversion: $x \rightarrow -x$. It follows that the existence of a solution $a_0(x)$, $f(x)$, and $s(x)$ of system (34)–(36) also means the existence of a solution $a_0(-x)$, $f(-x)$, and $s(-x)$. This fact, however, does not mean that $a_0(x)$, $f(x)$, and $s(x)$ must be even functions of x . Indeed, we shall see later that system (34)–(36) together with boundary conditions (39) has nonsymmetric soliton solutions.

Substituting the power expansions for the functions $a_0(x)$, $f(x)$, and $s(x)$ into Eqs. (34)–(36), we obtain the asymptotic form of the solution as $x \rightarrow 0$,

$$a_0(x) = a_0 + a_1 x + \frac{a_2}{2!} x^2 + O(x^3), \quad (43a)$$

$$f_0(x) = f_0 + f_1 x + \frac{f_2}{2!} x^2 + O(x^3), \quad (43b)$$

$$s_0(x) = s_0 + s_1 x + \frac{s_2}{2!} x^2 + O(x^3), \quad (43c)$$

where the next-to-leading coefficients

$$a_2 = 2a_0(e^2 f_0^2 + q^2 s_0^2) - 2q\omega s_0^2, \quad (44a)$$

$$f_2 = f_0(m_\phi^2 - g_\phi f_0^2 + h_\phi f_0^4 - e^2 a_0^2 + \lambda s_0^2), \quad (44b)$$

$$s_2 = s_0(m_\chi^2 - (\omega - qa_0)^2 - g_\chi s_0^2 + h_\chi s_0^4 + \lambda f_0^2) \quad (44c)$$

are determined in terms of the three leading coefficients a_0 , f_0 , and s_0 and the model's parameters. The next coefficients a_n , f_n , and s_n , where $n = 3, 4, 5, \dots$, are determined by the six leading coefficients a_0, f_0, s_0, a_1, f_1 , and s_1 and the model's parameters. It can be easily shown that if the coefficients a_1, f_1 , and s_1 vanish all the other coefficients with an odd n also vanish, and we have an even solution of Eqs. (34)–(36).

Linearization of Eqs. (34)–(36) at large x together with corresponding boundary conditions (39) leads us to the asymptotic form of the solution as $x \rightarrow \pm\infty$,

$$f(x) \sim f_{\pm\infty} \exp(\mp \tilde{m}_{\phi\pm} x), \quad (45a)$$

$$s(x) \sim s_{\pm\infty} \exp(\mp \tilde{m}_{\chi\pm} x), \quad (45b)$$

$$a_0(x) \sim a_{\pm\infty} + a_{\pm\infty} \frac{e^2 f_{\pm\infty}^2}{2\tilde{m}_{\phi\pm}^2} \times \exp(\mp 2\tilde{m}_{\phi\pm} x) - (\omega - qa_{\pm\infty}) \times \frac{qs_{\pm\infty}^2}{2\tilde{m}_{\chi\pm}^2} \exp(\mp 2\tilde{m}_{\chi\pm} x), \quad (45c)$$

where the mass parameters $\tilde{m}_{\phi\pm}$ and $\tilde{m}_{\chi\pm}$ are defined by the relations

$$\tilde{m}_{\phi\pm}^2 = m_\phi^2 - e^2 a_{\pm\infty}^2, \quad (46)$$

$$\tilde{m}_{\chi\pm}^2 = m_\chi^2 - (\omega - qa_{\pm\infty})^2. \quad (47)$$

From Eqs. (46) and (47), we obtain the boundaries on the absolute values of $a_0(\pm\infty) = a_{\pm\infty}$ and ω :

$$|a_0(\pm\infty)| < \frac{m_\phi}{e}, \quad |m_\chi - \frac{q}{e} m_\phi| < |\omega| < m_\chi + \frac{q}{e} m_\phi. \quad (48)$$

We shall see later that the upper boundaries for $|a_0(\pm\infty)|$ and $|\omega|$ are attained in the thick-wall regime, whereas the minimal values of $|\omega|$, which we are able to reach numerically in the thin-wall regime, are always more than $|m_\chi - qe^{-1} m_\phi|$.

From Eqs. (43)–(45), it follows that there may be two types of solutions: the symmetric one for which $f(-x) = f(x)$, $s(-x) = s(x)$, and $a_0(-x) = a_0(x)$ and the nonsymmetric one that does not possess this property. For a symmetric solution, the series coefficients a_n , f_n , and s_n with an odd n vanish, and so in Eqs. (45)–(47), the asymptotic parameters corresponding to $x \rightarrow -\infty$ are equal to those corresponding to $x \rightarrow \infty$.

If the values of the model's parameters are fixed, then the behavior of a nonsymmetric solution $f(x)$, $s(x)$, and $a_0(x)$ as $x \rightarrow 0$ is determined by the six parameters a_0, f_0, s_0, a_1, f_1 , and s_1 in Eqs. (43). The behavior of the nonsymmetric solution as $x \rightarrow \pm\infty$ is also determined by the six

parameters in Eqs. (45), namely, $a_{-\infty}$, $f_{-\infty}$, and $s_{-\infty}$ as $x \rightarrow -\infty$ and a_{∞} , f_{∞} , and s_{∞} as $x \rightarrow \infty$. Thus, we have 12 free parameters in all. The continuity condition for $f(x)$, $s(x)$, and $a_0(x)$ and their derivatives $f'(x)$, $s'(x)$, and $a'_0(x)$ at arbitrary $x < 0$ give us six equations. A similar condition at arbitrary $x > 0$ provides us with another six equations. Therefore, we shall have 12 equations for determining the 12 parameters. According to Ref. [32], this fact is an argument in favor of the existence of the nonsymmetric solution for the boundary value problem in some range of the model's parameters. Of course, similar arguments can also be applied to a symmetric solution.

Any solution of field equations (10)–(12) is an extremum of the action $S = \int_{-\infty}^{\infty} \mathcal{L} dx dt$. At the same time, the Lagrangian density (1) does not depend on time in the case of field configurations (32). It follows that any solution of Eqs. (34)–(36), satisfying boundary conditions (39), is an extremum of the Lagrangian $L = \int_{-\infty}^{\infty} \mathcal{L} dx$. Let $a_0(x)$, $f(x)$, and $s(x)$ be a solution of system (34)–(36), satisfying boundary conditions (39). After the scale transformation of the solution's argument $x \rightarrow \lambda x$, the Lagrangian L becomes a function of the scale parameter λ . The function $L(\lambda)$ has an extremum at $\lambda = 1$, so its derivative with respect to λ vanishes at this point: $dL/d\lambda|_{\lambda=1} = 0$. From this equation, we obtain the virial relation for the soliton system,

$$E^{(E)} + E^{(P)} - E^{(G)} - E^{(T)} = 0, \quad (49)$$

where

$$E^{(E)} = \int_{-\infty}^{\infty} \frac{a_0'^2}{2} dx \quad (50)$$

is the electric field's energy,

$$E^{(G)} = \int_{-\infty}^{\infty} (f'^2 + s'^2) dx \quad (51)$$

is the gradient part of the soliton's energy,

$$E^{(T)} = \int_{-\infty}^{\infty} ((\omega - qa_0)^2 s^2 + e^2 a_0^2 f^2) dx \quad (52)$$

is the kinetic part of the soliton's energy, and

$$E^{(P)} = \int_{-\infty}^{\infty} (V(f) + U(s) + W(f, s)) dx \quad (53)$$

is the potential part of the soliton's energy.

The obvious equality $E = E^{(E)} + E^{(T)} + E^{(G)} + E^{(P)}$ and virial relation (49) lead to the following representations for the soliton system's energy:

$$E = 2(E^{(T)} + E^{(G)}), \quad (54)$$

$$E = 2(E^{(P)} + E^{(E)}). \quad (55)$$

Integrating the term $a_0'^2/2$ in Eq. (38) by parts and using Eqs. (34), (37), and (39), we obtain one more representation for the energy,

$$E = \frac{1}{2} \omega Q_{\chi} + E^{(G)} + E^{(P)}, \quad (56)$$

which, in turn, leads to the relation between the Noether charge Q_{χ} , the electric field's energy $E^{(E)}$, and the kinetic energy $E^{(T)}$:

$$\omega Q_{\chi} = 2(E^{(E)} + E^{(T)}). \quad (57)$$

If the electromagnetic coupling constants of the complex scalar fields ϕ and χ are the same ($e = q$), then the renormalizable interaction $\phi^2 \chi^{*2} + \phi^{*2} \chi^2$ may be added to interaction potential (5). In this case, the Lagrangian (1) is still invariant under local gauge transformations (7), but it is not invariant under two independent global gauge transformations (8), so the electric charges of ϕ and χ fields are not conserved separately. Thus, the model will possess only one conserved Noether charge Q_N , which is proportional to the total electric charge Q of a field configuration: $Q_N = e^{-1} Q$. Using the functional $F = E - \Omega Q_N$ and the Hamiltonian formalism, we come to the following time dependence of the fields: $\phi(x, t) = f(x) \exp(-i\Omega t)$, $\chi(x, t) = s(x) \exp(-i\Omega t)$, and $A_0(x, t) = a_0(x)$. Note that, unlike Eqs. (32), both scalar fields ϕ and χ have identical time dependence. In this case, Gauss's law can be written as

$$\xi''(x) - 2e^2(f^2(x) + s^2(x))\xi(x) = 0, \quad (58)$$

where $\xi(x) = \Omega - ea_0(x)$. For $E^{(E)}$ to be finite, $\xi(x)$ must tend to constants as $x \rightarrow \pm\infty$, so $\xi'(x) \rightarrow 0$, and the electric charge Q vanishes. But it is easily shown that any solution of Eq. (58) [with the exception of the trivial one $\xi(x) = 0$, which corresponds to the zero electric field] cannot satisfy these boundary conditions because of the positivity of the coefficient $2e^2(f^2(x) + s^2(x))$. Indeed, if $\xi(-\infty) > 0$ and $\xi'(-\infty) = 0$ or $\xi(+\infty) > 0$ and $\xi'(+\infty) = 0$, then $\xi(x)$ is concave for all x , while if $\xi(-\infty) < 0$ and $\xi'(-\infty) = 0$ or $\xi(+\infty) < 0$ and $\xi'(+\infty) = 0$, then the function $\xi(x)$ is convex for all x . In any case, one of the boundary conditions $\xi'(x) \rightarrow 0$ cannot be satisfied, so the electric charge Q is different from zero, and the electric field's energy $E^{(E)}$ is infinite. Hence, there is no nontopological soliton system with a nonzero electric field if the term $\phi^2 \chi^{*2} + \phi^{*2} \chi^2$ is added to interaction potential (5).

IV. THICK-WALL AND THIN-WALL REGIMES OF THE SOLITON SYSTEM

In this section, we research properties of the symmetric soliton solution in two extreme regimes. In the thick-wall regime, the mass parameters \tilde{m}_ϕ and \tilde{m}_χ tend to zero, leading to a spatial spreading of the soliton system. This fact and Eqs. (46) and (47) lead to the limiting values of the potential $a_0(\infty)$ and the phase frequency ω in the thick-wall regime:

$$|a_0(\infty)| = \frac{m_\phi}{e}, \quad \omega_{\text{tk}} = \text{sgn}(a_0(\infty)) \left(m_\chi + \frac{q}{e} m_\phi \right). \quad (59)$$

In the thick-wall regime, where $\tilde{m}_\phi^2 \approx \tilde{m}_\chi^2 \rightarrow 0$, we undertake the scale transformation of the fields and the x coordinate,

$$\begin{aligned} f(x) &= \Delta \bar{f}(\bar{x}), & s(x) &= \Delta \bar{s}(\bar{x}), \\ a_0(x) &= \frac{m_\phi}{e} + \frac{\Delta^2}{m_\phi^2} \bar{a}_0(\bar{x}), & x &= \Delta^{-1} \bar{x}, \end{aligned} \quad (60)$$

where the scale factor Δ is defined as

$$\begin{aligned} \Delta^2 &= m_\phi^2 - e^2 a_0^2(\infty) \approx m_\chi^2 - (\omega - q a_0(\infty))^2 \\ &\approx \kappa^2 (\omega_{\text{tk}}^2 - \omega^2). \end{aligned} \quad (61)$$

In Eq. (61), the factor κ is expressed in terms of the scalar particles' masses and the gauge coupling constants:

$$\kappa = e \left[\frac{m_\phi m_\chi}{(em_\phi + qm_\chi)(em_\chi + qm_\phi)} \right]^{\frac{1}{2}}. \quad (62)$$

Let us consider the functional $F = E - \omega Q_\chi$. From Eq. (33), it follows that this functional is related to the energy functional by means of Legendre transformation: $F(\omega) = E(Q_\chi) - \omega Q_\chi$. On field configuration (60), the functional F can be written as

$$F(\omega) = \Delta^3 \bar{F} + O(\Delta^5), \quad (63)$$

where the functional \bar{F} does not depend on ω :

$$\begin{aligned} \bar{F} &= \int_{-\infty}^{\infty} \left[\bar{f}'(\bar{x})^2 + \bar{s}'(\bar{x})^2 + \bar{f}(\bar{x})^2 + \bar{s}(\bar{x})^2 \right. \\ &\quad \left. - \frac{g_\phi}{2} \bar{f}(\bar{x})^4 - \frac{g_\chi}{2} \bar{s}(\bar{x})^4 + \lambda \bar{f}(\bar{x})^2 \bar{s}(\bar{x})^2 \right] d\bar{x}. \end{aligned} \quad (64)$$

In the thick-wall regime, the phase frequency ω tends to the limiting value ω_{tk} , so the parameter Δ vanishes, and it is possible to ignore the higher-order terms in Δ in Eq. (63). Using known properties of Legendre transformation, we obtain sequentially

$$Q_\chi(\omega) = -\frac{dF(\omega)}{d\omega} = 3\bar{F}\kappa^3 \omega (\omega_{\text{tk}}^2 - \omega^2)^{\frac{1}{2}}, \quad (65)$$

$$\begin{aligned} E(\omega) &= F(\omega) - \omega \frac{dF(\omega)}{d\omega} \\ &= \bar{F}\kappa^3 (2\omega^2 + \omega_{\text{tk}}^2) (\omega_{\text{tk}}^2 - \omega^2)^{\frac{1}{2}}. \end{aligned} \quad (66)$$

From Eqs. (65) and (66), we obtain the dependence of the energy E on the Noether charge Q_χ in the thick-wall regime:

$$E(Q_\chi) = \omega_{\text{tk}} Q_\chi - \frac{1}{54} \frac{1}{\bar{F}^2 \kappa^6 \omega_{\text{tk}}^3} Q_\chi^3 + O(Q_\chi^5). \quad (67)$$

We see from Eqs. (20), (65)–(67) that the energy E and the Noether charges Q_ϕ and Q_χ of the soliton system tend to zero in the thick-wall regime. Further, Eq. (67), basic relation (33), and the inequality $\omega^2 < \omega_{\text{tk}}^2$ lead to the conclusion that $E(Q_\chi) < \omega_{\text{tk}} Q_\chi$ for all values of Q_χ . From Eqs. (20), (42), and (59), it follows that $\omega_{\text{tk}} Q_\chi$ is equal to $m_\phi |Q_\phi| + m_\chi |Q_\chi|$, which, in turn, is the rest energy of the neutral plane-wave configuration formed from the charged scalar ϕ and χ particles. Hence, the symmetric soliton system is stable against decay into the scalar ϕ and χ particles.

The second extremal regime of the symmetric soliton system is the thin-wall regime in which the absolute value of the phase frequency tends to some minimum value ω_{tn} . In the thin-wall regime, the spatial size of the soliton system increases indefinitely, with the result that its energy E and Noether charges Q_ϕ and Q_χ also tend to infinity. In the thin-wall regime, when the spatial size of the soliton system $L \rightarrow \infty$, the gradient operator gives a factor proportional to L^{-1} . Therefore, we can ignore electric field's energy (50) and gradient energy (51) in comparison with kinetic energy (52) and potential energy (53). Then, from Eq. (49), it follows that the limiting relation

$$\lim_{\omega \rightarrow \omega_{\text{tn}}} \frac{E^{(T)}}{E^{(P)}} = 1 \quad (68)$$

holds in the thin-wall regime and, as a consequence,

$$\lim_{\omega \rightarrow \omega_{\text{tn}}} \frac{2E^{(T)}}{E} = \lim_{\omega \rightarrow \omega_{\text{tn}}} \frac{2E^{(P)}}{E} = 1. \quad (69)$$

Further, electric charge density (37) tends to zero in the thin-wall regime, since the soliton system's electric charge is strictly equal to zero, whereas its spatial size tends to infinity. Then, using Eqs. (37), (52), and (69), we obtain the limiting relation

$$\lim_{\omega \rightarrow \omega_{\text{in}}} \frac{2E^{(T)}}{Q_\chi} = \lim_{\omega \rightarrow \omega_{\text{in}}} \frac{E}{Q_\chi} = \omega_{\text{in}}, \quad (70)$$

which is consistent with basic relation (33) and Eq. (57).

V. NUMERICAL RESULTS

The system of differential equations (34)–(36) with boundary conditions (39) is the mixed boundary value problem on the infinite interval $x \in (-\infty, \infty)$. This boundary value problem can be solved only by numerical methods. In this paper, the boundary value problem was solved using the MAPLE package [33] by the method of finite differences and subsequent Newtonian iterations. Equations (33), (20), and (49) were used to check the correctness of numerical solutions.

Let us discuss possible types of solutions of the boundary value problem. If the quartic coupling constant λ and the electromagnetic coupling constants e and q are set equal to zero, then the Lagrangian (1) will describe the system of two self-interacting complex scalar fields that, however, do not interact with each other. In this case, the boundary value problem has the solution describing a system of two noninteracting nongauged one-dimensional Q -balls. Generally, these two Q -balls have different shapes and can be at an arbitrary distance from each other, so the solution will not be symmetric. However, the situation changes when the electromagnetic interaction is turned on. In this case, from Eq. (20), it follows that the electric charges of two Q -ball components are equal in magnitude but opposite in sign. It is important to note that the electric charges of two gauged Q -balls are conserved separately owing to the neutrality of the Abelian gauge field. Since the opposite electric charges attract each other, the initially nonsymmetric soliton system transitions into a symmetric one. Now, we turn on the quartic interaction between the two complex scalar fields ϕ and χ by letting the coupling constant λ be some positive value. From Eq. (5), it follows that the energy of the quartic interaction increases with the increase of overlap between the Q -ball components of the soliton system and is negligible at large separations between the Q -ball components. Such a behavior of the quartic interaction corresponds to a short-range repulsive force between the Q -ball components, while the electromagnetic long-range attractive force results in the confinement of the Q -ball components. One would expect that for a sufficiently large positive coupling constant λ the action of these opposite forces leads to an equilibrium nonsymmetric soliton configuration, which is the solution of boundary value problem (34)–(36), and (39). Indeed, we shall see later that such a nonsymmetric soliton solution really exists.

The system of differential equations (34)–(36) depends on the ten dimensional parameters: ω , e , q , m_ϕ , m_χ , g_ϕ , g_χ , h_ϕ , h_χ , and λ . It is readily seen, however, that the

dimensionless functions $a_0(x)$, $f(x)$, and $s(x)$ can depend only on nine independent dimensionless combinations of these parameters. Therefore, without loss of generality, we can choose the mass m_ϕ of the scalar ϕ particle as the energy unit. We consider a general case in which the corresponding dimensionless parameters are values of the same order: $\tilde{e} = e/m_\phi = 0.2$, $\tilde{q} = q/m_\phi = 0.2$, $\tilde{m}_\chi = m_\chi/m_\phi = 1.25$, $\tilde{g}_\phi = g_\phi/m_\phi^2 = 1$, $\tilde{g}_\chi = g_\chi/m_\phi^2 = 1.5$, $\tilde{h}_\phi = h_\phi/m_\phi^2 = 0.22$, $\tilde{h}_\chi = h_\chi/m_\phi^2 = 0.31$, and $\tilde{\lambda} = \lambda/m_\phi^2 = 0.2$.

Figures 1 and 2 present the dependences of the soliton's dimensionless energy $\tilde{E} = m_\phi^{-1}E$ and Noether charge Q_χ on the dimensionless phase frequency $\tilde{\omega} = m_\phi^{-1}\omega$.

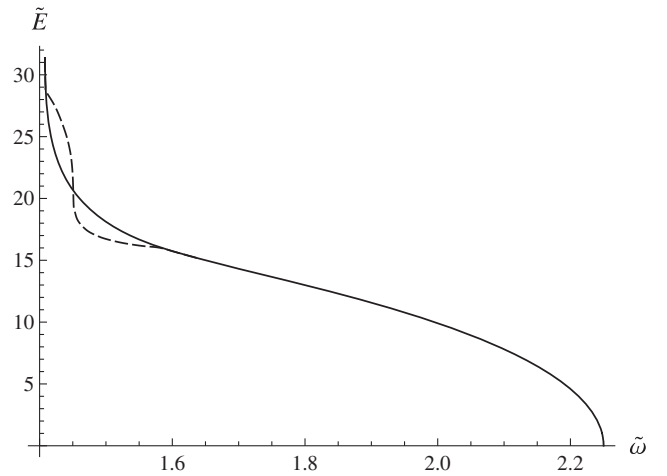


FIG. 1. The dependence of the dimensionless soliton energy $\tilde{E} = m_\phi^{-1}E$ on the dimensionless phase frequency $\tilde{\omega} = m_\phi^{-1}\omega$. The solid curve corresponds to the symmetric soliton system, and the dashed curve corresponds to the nonsymmetric one.

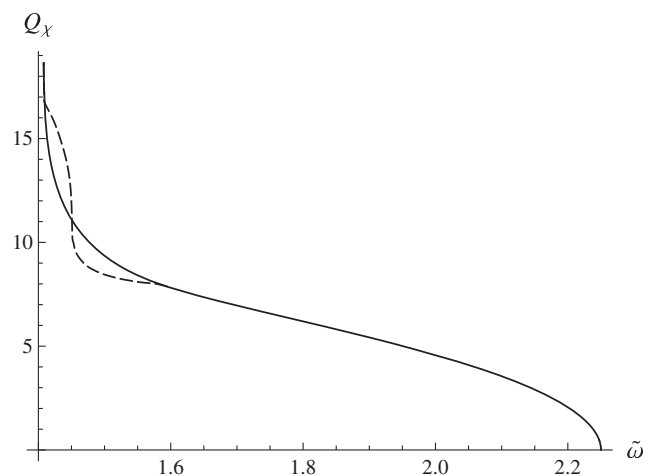


FIG. 2. The dependence of the soliton Noether charge Q_χ on the dimensionless phase frequency $\tilde{\omega} = m_\phi^{-1}\omega$. The solid curve corresponds to the symmetric soliton system, and the dashed curve corresponds to the nonsymmetric one.

The most striking feature of Figs. 1 and 2 is the coexistence of the symmetric and nonsymmetric soliton solutions. Indeed, it has been found numerically that the symmetric soliton solution exists in the range from the minimum value $\tilde{\omega}_{\min} = 1.4079916$, which we managed to reach by numerical methods, to the maximum value $\tilde{\omega}_{\text{tk}} = 2.25$. On the contrary, the nonsymmetric soliton solution exists only in the interval from the left bifurcation point $\tilde{\omega}_{\text{lb}} = 1.409$ to the right one $\tilde{\omega}_{\text{rb}} = 1.601$. We also see that the two types of curves have intersection points at $\tilde{\omega}_{i1}$ and $\tilde{\omega}_{i2}$ in Figs. 1 and 2, respectively. These intersection points are slightly different: $\tilde{\omega}_{i1} = 1.4504025$, whereas $\tilde{\omega}_{i2} = 1.4504280$. In each of the figures, the solid and dashed curves bound the two regions, which connect at the intersection points. Using Eq. (33), it can easily be shown that the areas of these regions are equal to each other, so we have the relations

$$\int_{\tilde{\omega}_{\text{lb}}}^{\tilde{\omega}_{\text{rb}}} [Q_{\chi^a}(\tilde{\omega}) - Q_{\chi^s}(\tilde{\omega})] d\tilde{\omega} = 0 \quad (71)$$

and

$$\int_{\tilde{\omega}_{\text{lb}}}^{\tilde{\omega}_{\text{rb}}} [\tilde{E}_a(\tilde{\omega}) - \tilde{E}_s(\tilde{\omega})] d\tilde{\omega} = 0, \quad (72)$$

which were checked numerically.

When $\tilde{\omega}$ tends to its minimal value $\tilde{\omega}_{\text{tn}}$, the symmetric soliton system goes into the thin-wall regime. In this regime, the energy \tilde{E} , the Noether charges Q_χ and Q_ϕ , and the effective spatial size L of the symmetric soliton system increase indefinitely. In particular, we found numerically that \tilde{E} , Q_χ , Q_ϕ , and L increase logarithmically as $\tilde{\omega} \rightarrow \tilde{\omega}_{\text{tn}}$,

$$\tilde{E} \sim -\tilde{\omega}_{\text{tn}} B \ln(\tilde{\omega} - \tilde{\omega}_{\text{tn}}), \quad (73)$$

$$Q_\chi \sim -B \ln(\tilde{\omega} - \tilde{\omega}_{\text{tn}}), \quad (74)$$

$$Q_\phi \sim B \frac{q}{e} \ln(\tilde{\omega} - \tilde{\omega}_{\text{tn}}), \quad (75)$$

$$L \sim -C \ln(\tilde{\omega} - \tilde{\omega}_{\text{tn}}), \quad (76)$$

where B and C are some positive constants, and the limiting thin-wall phase frequency $\tilde{\omega}_{\text{tn}} = 1.4079869$. Note that this numerical estimation of $\tilde{\omega}_{\text{tn}}$ is slightly less than the minimal value $\tilde{\omega}_{\min} = 1.4079916$, which was reached by numerical methods. Note also that in the thin-wall regime the behavior of E , Q_χ , Q_ϕ , and L is similar to that of the corresponding values of the one-dimensional nongauged Q -ball, as follows from Eqs. (73)–(76) and (A9)–(A11).

When $\tilde{\omega}$ tends to its maximal value $\tilde{\omega}_{\text{tk}}$, the symmetric soliton system goes into the thick-wall regime. In this regime, the soliton system is spread out over one-dimensional space, while the amplitudes of the scalar fields ϕ and χ tend to

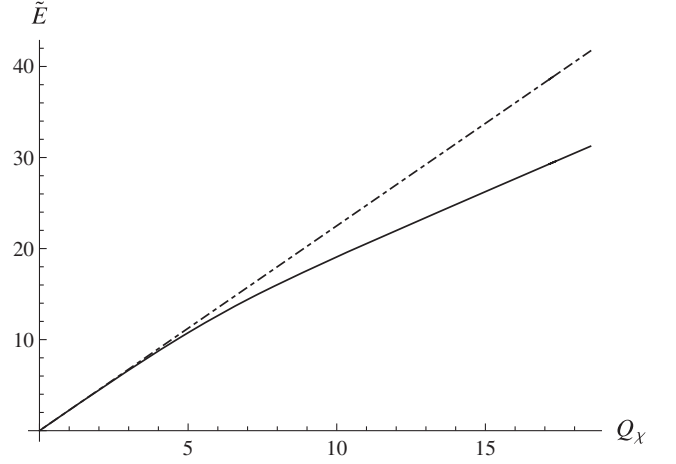


FIG. 3. The dependence of the dimensionless energy $\tilde{E} = m_\phi^{-1}E$ of the symmetric soliton system on the Noether charge Q_χ (solid curve). The dash-dotted line is the straight line $\tilde{E} = \tilde{\omega}_{\text{tk}}Q_\chi = (1 + m_\chi/m_\phi)Q_\chi$.

zero as $(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{1/2}$ in accordance with Sec. IV. It was found numerically that in the thick-wall regime \tilde{E} , Q_χ , and Q_ϕ also tend to zero as $(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{1/2}$, whereas the effective spatial size L diverges as $(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{-1/2}$:

$$\tilde{E} \sim b\tilde{\omega}_{\text{tk}}(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{1/2}, \quad (77)$$

$$Q_\chi \sim b(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{1/2}, \quad (78)$$

$$Q_\phi \sim -b \frac{q}{e} (\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{1/2}, \quad (79)$$

$$L \sim c(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{-1/2}. \quad (80)$$

From Eqs. (77)–(80) and (A5)–(A7), it follows that the behavior of E , Q_χ , Q_ϕ , and L is similar to that of the corresponding values of the one-dimensional nongauged Q -ball in the thick-wall regime.

Figure 3 shows the dependence of the dimensionless energy \tilde{E} of the symmetric soliton system on the Noether charge Q_χ . We see that the dependence $\tilde{E}(Q_\chi)$ is an increasing convex ($d\tilde{E}/dQ_\chi > 0$, $d^2\tilde{E}/dQ_\chi^2 < 0$) function outgoing from the coordinate origin. It follows that the symmetric soliton system is stable against decay into several smaller ones. We also see that, in accordance with Sec. IV, the curve $\tilde{E}(Q_\chi)$ lies below the straight line $\tilde{E} = \tilde{\omega}_{\text{tk}}Q_\chi$ for all positive Q_χ . From this, it follows that the symmetric soliton system is stable against decay into massive scalar ϕ and χ bosons.

In Fig. 4, we can see the dependence of the energy difference $\Delta\tilde{E} = \tilde{E}_s - \tilde{E}_a$ between the symmetric and nonsymmetric soliton solutions on the Noether charge Q_χ . From Fig. 4, it follows that the energy of the symmetric

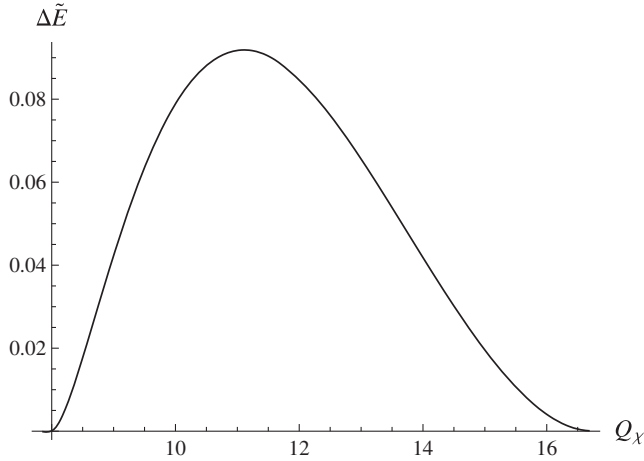


FIG. 4. The dependence of the energy difference $\Delta\tilde{E} = \tilde{E}_s - \tilde{E}_a$ between the symmetric and nonsymmetric soliton solutions on the Noether charge Q_χ .

soliton system slightly exceeds the energy of the nonsymmetric one in the whole range of the Noether charge Q_χ for which the existence of the nonsymmetric soliton system is possible. It follows that the nonsymmetric soliton system is more preferable from the viewpoint of energy as compared to the symmetric one. Note, however, that the difference $\Delta\tilde{E}$ is rather small and is of the order of 0.1% of the soliton system’s energy. Just as the symmetric soliton system, the nonsymmetric one is stable against decay into massive scalar bosons. Note, however, that the symmetric soliton system may transition into the nonsymmetric one through quantum tunneling.

Figure 5 presents the nonsymmetric soliton solution corresponding to the dimensionless phase frequency $\tilde{\omega} = 1.5$, whereas Fig. 6 presents the energy and electric charge densities and the electric field strength corresponding to Fig. 5. The nonsymmetric character of the soliton system is

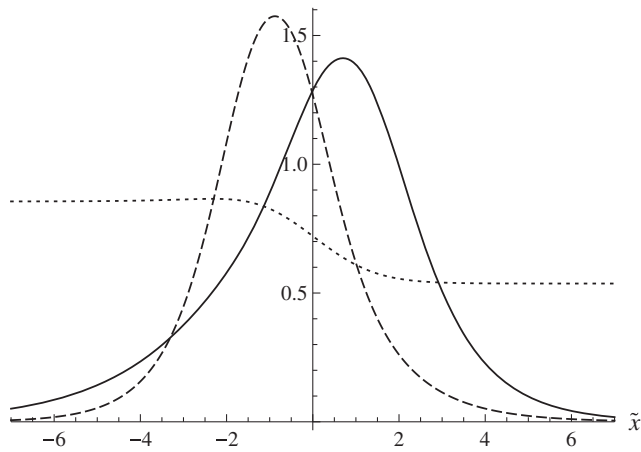


FIG. 5. The nonsymmetric numerical solution for $f(\tilde{x})$ (solid curve), $s(\tilde{x})$ (dashed curve), and $\tilde{e}a_0(\tilde{x})$ (dotted curve). The dimensionless phase frequency $\tilde{\omega} = 1.5$.

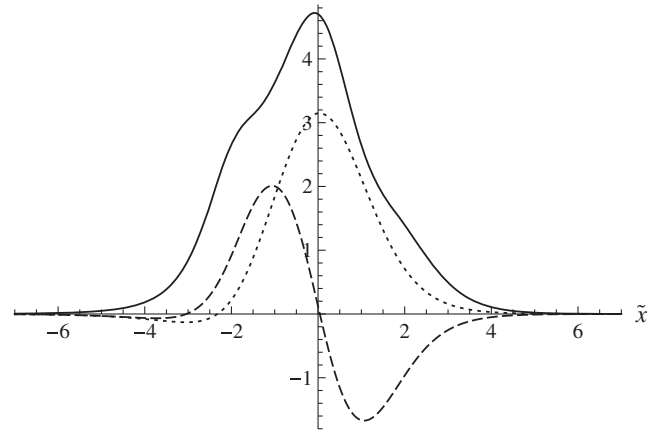


FIG. 6. The dimensionless versions of the energy density $\tilde{\mathcal{E}} = m_\phi^{-2}\mathcal{E}$ (solid curve), the scaled electric charge density $\tilde{e}^{-1}\tilde{j}_0 = \tilde{e}^{-1}m_\phi^{-2}j_0$ (dashed curve), and the scaled electric field strength $\tilde{e}^{-1}\tilde{E}_x = \tilde{e}^{-1}m_\phi^{-1}E_x$ (dotted curve), corresponding to the nonsymmetric solution in Fig. 5.

obvious from Figs. 5 and 6. The most interesting feature of the nonsymmetric soliton system is the presence of the unidirectional electric field in its interior, as for a plane capacitor. From Fig. 5, it follows that the charged scalar ϕ and χ particles can acquire the energy equal to $-e\Delta a_0 \approx 0.32m_\phi$ in the electric field of the nonsymmetric soliton system. Note that this energy is comparable with the scalar particles’ masses. Lighter particles (e.g., light charged fermions) passing through the interior of the nonsymmetric soliton system can be accelerated to relativistic velocities and energies.

Similar to Figs. 5 and 6, Figs. 7 and 8 give information about the symmetric soliton solution. From Fig. 8, it follows that the energy and electric charge densities are symmetric with respect to the center of the soliton system,

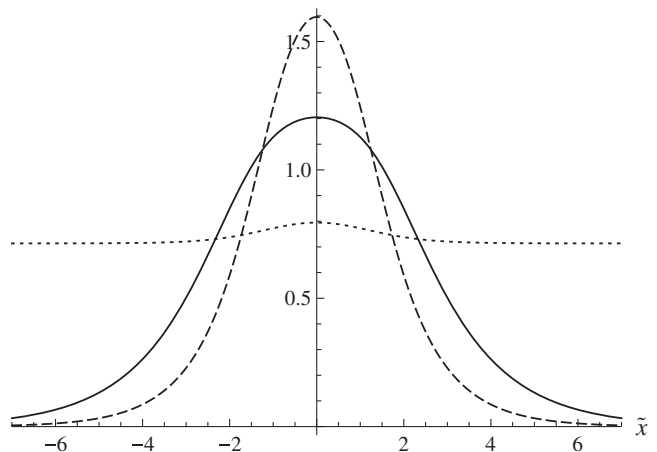


FIG. 7. The symmetric numerical solution for $f(\tilde{x})$ (solid curve), $s(\tilde{x})$ (dashed curve), and $\tilde{e}a_0(\tilde{x})$ (dotted curve). The dimensionless phase frequency $\tilde{\omega} = 1.5$.

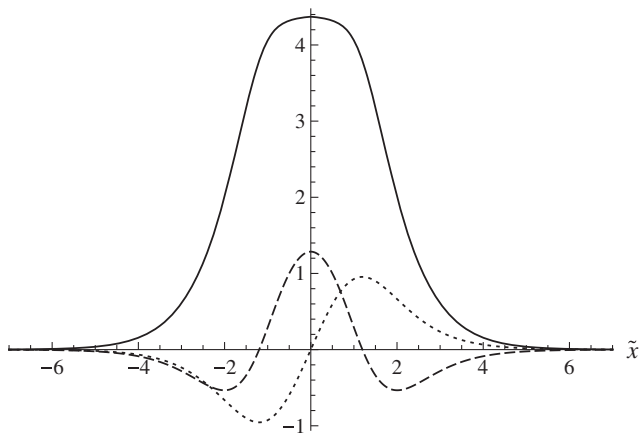


FIG. 8. The dimensionless versions of the energy density $\tilde{\mathcal{E}} = m_\phi^{-2}\mathcal{E}$ (solid curve), the scaled electric charge density $\tilde{e}^{-1}\tilde{j}_0 = \tilde{e}^{-1}m_\phi^{-2}j_0$ (dashed curve), and the scaled electric field strength $\tilde{e}^{-1}\tilde{E}_x = \tilde{e}^{-1}m_\phi^{-1}E_x$ (dotted curve), corresponding to the symmetric solution in Fig. 7.

while the electric field strength is antisymmetric. For positive ω , it is directed from the soliton system's center, so it attracts negatively charged particles and repels positively charged ones. For negative ω , it is directed to the soliton system's center, so the roles of negatively and positively charged particles are interchanged. For positive (negative) ω , the form of the electromagnetic potential a_0 corresponds to a potential well for negatively (positively) charged particles. It follows that bound fermionic and bosonic states can exist in the electric field of the symmetric soliton system.

VI. CONCLUSION

In the present paper, the one-dimensional nontopological soliton system consisting of two self-interacting complex scalar fields has been investigated. The scalar fields interact with each other through the Abelian gauge field and the quartic scalar interaction. The finiteness of the energy of the one-dimensional soliton system leads to its electric neutrality, so its two scalar components have opposite electric charges. The neutrality of the Abelian gauge field leads to the separate conservation of the electric charges of these scalar components. The interplay between the attractive electromagnetic interaction and the repulsive quartic interaction leads to the existence of symmetric and nonsymmetric soliton systems.

The symmetric soliton system exists in the whole allowable interval of the phase frequency ω . When ω tends to its minimal (maximal) value, the symmetric soliton system goes into the thin-wall (thick-wall) regime. In the thin-wall regime, the energy, the Noether charges, and the spatial size of the symmetric soliton system tend to infinity. In the thick-wall regime, the spatial size of the symmetric soliton system also tends to infinity, but the energy and the

Noether charges tend to zero. In contrast to this, the nonsymmetric soliton system exists only in some interior subinterval between the minimal and maximal allowable phase frequencies ω_{tn} and ω_{tk} . It follows that there exists an interval of the Noether charge Q_χ (and, consequently, an interval of the Noether charge $Q_\phi = -qe^{-1}Q_\chi$), where the symmetric and nonsymmetric soliton systems coexist. In all of this interval, the energy of the nonsymmetric soliton system turns out to be less than that of the symmetric soliton system, so the symmetric soliton system can turn into the nonsymmetric one through quantum tunneling. Both symmetric and nonsymmetric soliton systems are stable against decay into massive scalar ϕ and χ bosons.

Despite the fact that the soliton system is electrically neutral, it nevertheless possesses a nonzero electric field in its interior. Note that the electric fields of the symmetric and nonsymmetric soliton systems are essentially different. The electric field of the nonsymmetric soliton system is unidirectional in its interior, like the electric field of a plane capacitor. It can accelerate light particles up to relativistic velocities and energies. In contrast, the electric field of the symmetric soliton system corresponds to the electromagnetic potential of a potential well. In such an electric field, the existence of bound bosonic and fermionic states is possible.

It is known [1,16] that the field configuration of a nontopological soliton composed only of scalar fields can be described in terms of a mechanical analogy. For the one-dimensional case, it corresponds to the motion of a particle with the unit mass in the time x in the conservative force field of a certain potential. The dimension of space in which the particle moves is equal to the number of scalar fields constituting the nontopological soliton. Using this analogy, one can easily explain the behavior of the pure scalar nontopological soliton in both the thin-wall and the thick-wall regimes. One can also determine whether a soliton solution exists for a given set of the model's parameters. At the same time, the system of differential equations (34)–(36) describing the soliton system of the present paper has no interpretation in terms of any mechanical analogy. For this reason, the existence of the soliton system should be established numerically for any given set of the model's parameters.

Finally, let us stress the specific character of the $(1+1)$ -dimensional electromagnetic field. Its characteristic feature is the absence of nondiagonal terms of the electromagnetic stress-energy tensor. This is because the magnetic field does not exist in $(1+1)$ dimensions, so the Poynting vector vanishes there. Therefore, the $(1+1)$ -dimensional electromagnetic field cannot transfer any energy or momentum. Instead, the scalar fields' kinetic energy can transform to the one-dimensional electric field's energy, which, in turn, can transform back to the scalar fields' energy. Note also that in $(1+1)$ dimensions, the potential energy of two oppositely charged particles is proportional to the distance

between them, so the electromagnetic interaction has confining character. Thus, we can conclude that the $(1+1)$ -dimensional electromagnetic interaction is similar to an elastic string. The only difference is that there is no energy nor momentum transfer in the one-dimensional electric field, whereas in the elastic string, waves can transfer energy and momentum. The behavior of the $(1+1)$ -dimensional electromagnetic field is completely determined by Gauss's law, which is not a dynamic field equation but is the condition imposed on an initial field configuration. Indeed, in the adopted gauge $A_x = 0$, Gauss's law does not contain time derivatives of the electromagnetic potential A_0 . In this connection, it can be said that the $(1+1)$ -dimensional electromagnetic field is not a dynamic one.

ACKNOWLEDGMENTS

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APPENDIX: THE ONE-DIMENSIONAL NONGAUGED Q -BALL

Here, we collect formulas concerning the one-dimensional nongauged Q -ball in the model of a self-interacting complex scalar field with the six-order self-interaction potential $V(|\phi|) = m^2|\phi|^2 - g|\phi|^4/2 + h|\phi|^6/3$. The analytical Q -ball solution exists only in the $(1+1)$ dimensions [1], where it can be written as

$$\begin{aligned} \phi(t, x) = & \frac{2}{\sqrt{g}} \sqrt{m^2 - \omega^2} \left(1 + \left(1 - \frac{m^2 - \omega^2}{m^2 - \omega_{\text{tn}}^2} \right)^{\frac{1}{2}} \right. \\ & \times \cosh \left(2\sqrt{m^2 - \omega^2} (x - x_0) \right) \left. \right)^{-\frac{1}{2}} \\ & \times \exp(-i\omega(t - t_0)). \end{aligned} \quad (\text{A1})$$

In Eq. (A1), the squared phase frequency $\omega^2 \in (\omega_{\text{tn}}^2, m^2)$, where

$$\omega_{\text{tn}}^2 = m^2 \left(1 - \frac{3}{16} \frac{g^2}{m^2 h} \right). \quad (\text{A2})$$

The Noether charge and the energy of the one-dimensional Q -ball can be expressed in a rather compact form:

$$Q = 4\omega \sqrt{\frac{3}{h}} \operatorname{arctanh} \left(\left[\frac{m^2 - \omega_{\text{tn}}^2}{m^2 - \omega^2} \right]^{\frac{1}{2}} - \left[\frac{m^2 - \omega_{\text{tn}}^2}{m^2 - \omega^2} - 1 \right]^{\frac{1}{2}} \right), \quad (\text{A3})$$

and

$$\begin{aligned} E = & \omega Q - 2\sqrt{\frac{3}{h}} (\omega^2 - \omega_{\text{tn}}^2) \\ & \times \operatorname{arctanh} \left(\left[\frac{\sqrt{m^2 - \omega_{\text{tn}}^2} - \sqrt{\omega^2 - \omega_{\text{tn}}^2}}{\sqrt{m^2 - \omega_{\text{tn}}^2} + \sqrt{\omega^2 - \omega_{\text{tn}}^2}} \right]^{\frac{1}{2}} \right) \\ & + \sqrt{\frac{3}{h}} \sqrt{(m^2 - \omega^2)(m^2 - \omega_{\text{tn}}^2)}. \end{aligned} \quad (\text{A4})$$

Let us present the expressions of the Noether charge Q and the energy E in two extreme regimes. In the thick-wall regime, the squared phase frequency tends to its maximum value: $\omega^2 \rightarrow m^2$. Using Eqs. (A3) and (A4), we obtain the expressions of the soliton's Noether charge and energy in the thick-wall regime,

$$\begin{aligned} Q = & \operatorname{sgn}(\omega) \frac{2\sqrt{6}m^{3/2}}{\sqrt{h(m^2 - \omega_{\text{tn}}^2)}} \sqrt{\delta} \\ & \times \left(1 - \frac{7m^2 - 15\omega_{\text{tn}}^2}{12m(m^2 - \omega_{\text{tn}}^2)} \delta + O(\delta^2) \right), \end{aligned} \quad (\text{A5})$$

and

$$\begin{aligned} E = & \frac{2\sqrt{6}m^{5/2}}{\sqrt{h(m^2 - \omega_{\text{tn}}^2)}} \sqrt{\delta} \\ & \times \left(1 - \frac{11m^2 - 19\omega_{\text{tn}}^2}{12m(m^2 - \omega_{\text{tn}}^2)} \delta + O(\delta^2) \right), \end{aligned} \quad (\text{A6})$$

where the variable $\delta = m - |\omega|$. Furthermore, Eq. (A1) leads to the soliton's width at half-height in the thick-wall regime:

$$\begin{aligned} L = & \frac{\cosh^{-1}(7)}{\sqrt{2}} \frac{1}{\sqrt{m\delta}} + \frac{1}{4\sqrt{2}} \\ & \times \left(\frac{\sqrt{3m}}{m^2 - \omega_{\text{tn}}^2} + \frac{\cosh^{-1}(7)}{m^{3/2}} \right) \sqrt{\delta} + O(\delta^{\frac{3}{2}}). \end{aligned} \quad (\text{A7})$$

Using Eqs. (A5) and (A6), we obtain the dependence of E on Q in the thick-wall regime:

$$E = m|Q| - \frac{h}{3!} \frac{m^2 - \omega_{\text{tn}}^2}{12m^3} |Q|^3 + O(|Q|^5). \quad (\text{A8})$$

From Eqs. (A5)–(A7), it follows that in the thick-wall regime the soliton's Noether charge and energy vanish as $\sqrt{\delta}$, whereas the soliton's effective size diverges as $1/\sqrt{\delta}$.

In the thin-wall regime, the squared phase frequency tends to its minimum value: $\omega^2 \rightarrow \omega_{\text{tn}}^2$. In this regime, the Noether charge, the energy, and the width at half-height of the one-dimensional Q -ball behave as

$$Q = \operatorname{sgn}(\omega) \sqrt{\frac{3}{h}} \omega_{\text{tn}} \left[\ln \left(\frac{2(m^2 - \omega_{\text{tn}}^2)}{\omega_{\text{tn}} \bar{\delta}} \right) - \sqrt{\frac{2\omega_{\text{tn}} \bar{\delta}}{m^2 - \omega_{\text{tn}}^2}} + O(\bar{\delta}) \right], \quad (\text{A9})$$

$$E = \sqrt{\frac{3}{h}} \omega_{\text{tn}}^2 \left[\ln \left(\frac{2(m^2 - \omega_{\text{tn}}^2)}{\omega_{\text{tn}} \bar{\delta}} \right) + \frac{m^2}{\omega_{\text{tn}}^2} - 1 - \sqrt{\frac{2\omega_{\text{tn}} \bar{\delta}}{m^2 - \omega_{\text{tn}}^2}} + O(\bar{\delta}) \right], \quad (\text{A10})$$

and

$$L = 2^{-1} (m^2 - \omega_{\text{tn}}^2)^{-1/2} \ln \left(\frac{18(m^2 - \omega_{\text{tn}}^2)}{\omega_{\text{tn}} \bar{\delta}} \right) + 4 \frac{\sqrt{2}}{3} \frac{\sqrt{\omega_{\text{tn}} \bar{\delta}}}{m^2 - \omega_{\text{tn}}^2} + O(\bar{\delta}), \quad (\text{A11})$$

where the variable $\bar{\delta} = |\omega| - \omega_{\text{tn}}$. From Eqs. (A9) and (A10), we obtain the dependence of E on Q in the thin-wall regime:

$$E = \omega_{\text{tn}} |Q| + \sqrt{\frac{3}{h}} (m^2 - \omega_{\text{tn}}^2) + O \left(\exp \left(-\sqrt{\frac{h}{3}} \frac{|Q|}{\omega_{\text{tn}}} \right) \right). \quad (\text{A12})$$

From Eqs. (A9)–(A11), it follows that the Noether charge, the energy, and the effective size of the one-dimensional Q -ball logarithmically diverge in the thin-wall regime.

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