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THEORIES OF «SMALL» AND «LARGE» CURVATURE OF BARS IN TOTAL MATHEMATICAL FORMULATION

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The theories of «small» и «large» displacements at rod crooking have been analysed with estimation and determination of assumption functions separating them. General mathematical maintenance on the basis of line curvature expressions in the parametric form is developed. Boundary value problem of rod curvature geometry is presented in two problems: «line reestablishment» by the curvature function and the initial conditions, then «line flattering» i.e. determination of curve arc length by the final conditions.

1. Introduction

Mechanics of deformable body deals with functional relations between the parameters characterising their states and the external influence. Its objectives consist, mainly, in establishing the changes in body geometry.

For most of the constructions the requirements for hardness limit the value of form and size transformations of the elements forming them and, correspondingly, by the notions of «small» and «large» the two approaches are formulated in the geometry of deformation. One distinguishes short (rigid) bars, whose physical resource of material resilience is exhausted at «small» changes in forms and sizes, and long (slender) ones, tolerating «large» changes in geometry at the same resource of resilience.

To define «small» changes a number of «guide rules and lines» (the insufficient changes in form, the rule of relative hardness, and the principle of unaltered initial sizes) has been formulated, they form a concept system of the «small displacements» or «small deformations» theory, methods and techniques of which represent the main content of «Strength of materials» course. Due to functional connection between the load and «characteristic displacement» in this theory there appears a division of the system into linear and nonlinear ones, there appears some terminological points: slightly curved axis of bar is called as deflection curve, «exact form of curve line» is called as elastics [1].

Approach to the problem on definition of deformation system geometry with long (slender) bars is characterised by assumption that for its solution «it is impossible to apply the conventional theory of material strength. One needs to design a completely different applied bending theory true for arbitrary large elastic displacement and differed fundamentally from the conventional theory beginning from the basic statements and concepts» [2].

The main equations of deformation mechanics of any form «have been reduced to the determining equations» long ago [3] and by now «the theory of large displacements, different from the conventional theory» exists [2], there are some investigations [1, 4], that are distinguished by complex transformations, reducing solution to special functions without obvious physical relation of their variables with investigated elastic parameters.

It has been noted [3] that «deformation mechanics consists not only of equations, but also of determinations of exact physical meaning of all parameters and functions included in these equations as well as equations themselves». Evidently, owing to the absence of these determinations the special theory without common grounds and obvious connections with the conventional theory has not become an engineering tool. In engineering education the approximate «theory of small displacements» prevails, but the results of separate problems solution on the special theory are used to prove the results of the approximate theory and to demonstrate nonlinear behaviour of some systems at «small» changes [5].

The concept of fundamental difference of «short» and «long» bars theory did not appear immediately [6]. Complexity of problem in its strict statement predetermined the theory of «small» displacement, but its effectiveness, meeting practical demands, moved aside from scientific interest and engineering requirements and prevented from development of their general theory to some extent.

Development of numerical methods and computer engineering has created a new way of thinking with the assumption that computerized analysis «with nonlinearity, which occurs in solving practical problems connected with construction design, does not result in insurmountable difficulties» [5]. Program complexes have been created that «can successfully be used for nonlinear problems». However, it should be noted that modern computer investigation is a multiple solution of linear problem. Process without physical content does not enable to establish single-value truth connections. Obviously, potentials of such analysis should be estimated from the viewpoint of its results. Undoubtedly, demands for value of intermediate and final results are different. For final results any form of presentation has an apparent value, for intermediate results analyticity (expression in elementary functions) or possibility of their mathematical manipulation is more important in our mind. This requirement is not a result of devotion to analytical expressions, i. e. a natural and necessary condition of physical interpretation of the phenomenon under study.

In mechanics of solid body deformation like in any science there are many blanks which, covered by the developed concepts, remain the same for a long time. S.P. Timoshenko pointed out [6] that «from time to time it is necessary to discuss the main assumptions, on which the methods of analysis are based». L.I. Sedov noted [7] that «apparent establishing of common bases and internal connections between different theories and the effects observed allow deep understanding of real scientific state, correct estimation of known and developed scientific achievements». Thus, the purpose of the work is to reveal common and internal connections between the two theories discussed.

2. Existent mathematical maintenance of the theory

For bars, elastically curved along plane curve, the problem of its assignment is formulated by the Bernoulli equation

$$K = M/EJ\,, \tag{1}$$

where K is the curvature, M is the bending moment, EJ is the bar rigidity.

Curvature characterises the measure of line curve and is defined as the rate of change in tangent inclination angle θ in the point at its motion along the curve L:

$$K = \frac{d\theta}{dL}. \tag{2}$$

For the line specified by the equation $y=f(x)$ curvature parameters (2), expressed through corresponding variables result in the known formula of mathematical analysis:

$$K = \frac{d^2y}{dx^2}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{-3/2}. \tag{3}$$

With the expressions (2) or (3) the equation (1) forms a nonlinear differential equation, by which the problem of elastics id correctly formulated.

In this statement the problem was studied by Y. Bernoulli, L. Euler, and J.L. Lagrange, J.R. Kirchhoff, A. Klebsh, B. Saint-Venant and many others. J.R. Kirchhoff by 1859 [6] had stated the identity of the equations for solid body rotation relative to fixed point and the equations for bar equilibrium deformed by forces applied to its ends. At present «Kirchhoff's dynamic analogy» forms the basis of «the theory of small displacement» presented by the methods [2], that found some application [8], but did not fit in traditional engineering courses.

The basis for the theory of «small displacements» is «the method of curve lines, having insufficient deviation from straight lines», suggested by L. Euler. The method consists in change of exact curvature expression (3) by simplified one:

$$K^* = d^2y/dx^2. \tag{4}$$

L. Euler did not estimate his suggestion mathematically. Up to nowadays it is believed [9] that «having confined himself by consideration of rather «small deformations», he decided to take arc differential dL approximately as abscissa differential dx and thus transformed the exact expression into approximate one». This statement in this or that form is hitherto proved in educational and engineering literature.

F.S. Yasinski [9], having expanded (3) in series,

$$K = \frac{d^2y}{dx^2}\left[1-\frac{3}{2}\left(\frac{dy}{dx}\right)^2+\frac{3\cdot 5}{2!\cdot 2^2}\left(\frac{dy}{dx}\right)^4+\cdots\right],$$

estimated (4) in terms of residual series

$$K^*-K \approx \frac{3}{2}\frac{d^2y}{dx^2}\,\mathrm{tg}^2\theta,$$

to show that the curvature expression suggested by him

$$K^*_я = d^2y/dL^2$$

is more exact.

This approximate estimation and current explanation of changes of (3) into (4) are not enough to have an idea of limitation on values of determined displacements and effects conditioned by the taken simplification. The terms «method of curve lines, having insufficient deviation from straight lines» or «method of slight displacements (deformations)» assume silently that value of displacements is determined in the coordinate system connected with initial position of non-deformed bar. However, curvature cannot be characterised by the value of linear displacements. Thus, for example, for the bar, curvature of which is determined by angular deflection of cross-sections, «large» displacements in x_1y_1 axis system can be «small» in x_2y_2 system (Fig. 1).

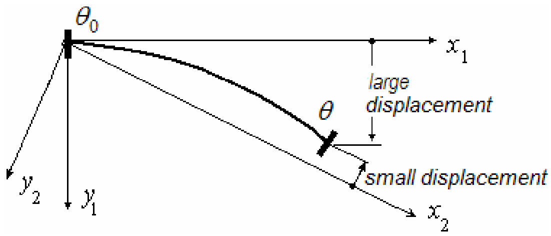

Fig. 1. *Displacements in curved bar*

Evidently, simplified curvature expression (4) allows the research in geometry of bar deformation with «large» displacements, if its changes do not go beyond displacements limits in other coordinate system. «Slightness of deformation» is also connected with displacement value; it is determined by the potential of material elasticity. Deformation value can be judged only angular deflection of cross-sections: they are the same in any coordinate system.

3. Estimation of simplified curvature expression

Introduction of the expressions (4) into analysis of deformation geometry is interpreted at apparent equivalence for small angles $\mathrm{tg}\theta\approx\sin\theta\approx\theta$ as an assumption of equality $dL=dx$ in (2) or neglect of dy/dx value square in comparison with a unit in (3). As arc differential of the curve line is connected with abscissa differential by the relationship $dx=dL.\cos\theta$, then the equality $dx=dL$ can be associated with assumptions

$$\cos\theta\approx 1 \text{ or } 1/\cos\theta\approx 1.$$

In fact, (4) follows from (3), if one takes

$$[1+(dy/dx)^2]^{-3/2}\approx 1.$$

Let us transform the right part, substituting the derivative in terms of its definition

$$\frac{dy}{dx} = \mathrm{tg}\theta, \quad [1+\mathrm{tg}^2\theta]^{-3/2} = \cos^3\theta.$$

We come to the conclusion that substitution of the expression (3) into (4) is equivalent to introduction of the assumption $\cos^3\theta=1$ or $1/\cos^3\theta=1$. These assumptions are cruder than it is suggested. There is a strict correspondence between the curvature expressions

$$K = K^*\cos^3\theta. \tag{5}$$

Application of simplified curvature expression K^* leads only to linearization of the differential equations (1), the assumption $dx=dL$ has got the other assignment.

4. Possible variants of curvature expression simplification

Simplified curvature expression K^* is formally obtained by division of its exact expression K by the function $\cos^3\theta$. This mathematical operation increases curvature and defines some line located from the side of its concavity with respect to the true one. With this function-assumption, applying multiplication, one can similarly define the line of less curvature situated from another side of the true line. Using weaker functions in significance $\cos^2\theta$, $\cos\theta$, without overstepping the limits of assumption introduced by L. Euler we obtain the curvature spectrum expression:

$$K^* = K\left(\frac{1}{\cos^3\theta} \cdots \cos^3\theta\right). \tag{6}$$

It includes the «refined» formula, suggested by F.S. Yasinski [9]

$$K^*_я = \frac{d^2y}{dL^2} = \frac{d}{dL}\left(\frac{dy}{dL}\right) = \frac{d}{(dx/\cos\theta)}\left(\frac{dL\cdot\sin\theta}{dL}\right) =$$

$$= \cos\theta\,\frac{d(\sin\theta)}{dx} = K\cos\theta.$$

One can say that F.S. Yasinski adopted a more strict assumption and with multiplication he obtained his curvature expression. L. Euler had chosen division before him and used the function $\cos^3\theta$.

Presentation of bar axis in pure bending (Fig. 2), demonstrates the effect of using different curvature expressions (6).

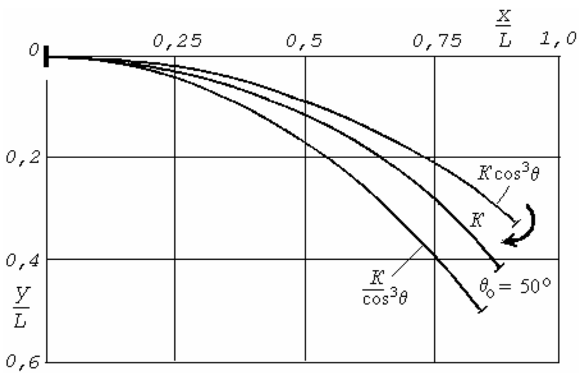

Fig. 2. *Bar curvature depending on curvature expression*

Fig. 3 shows the connection of stress and bending of displaced end and the boundary between «small» and «large» deformations, up to which the curvature expressions (6) are defined by the curves close to each other. Decreasing significance of simplifying function the differential equation becomes more exact, and that boundary (3 % in difference) is displaced to the direction of large displacements.

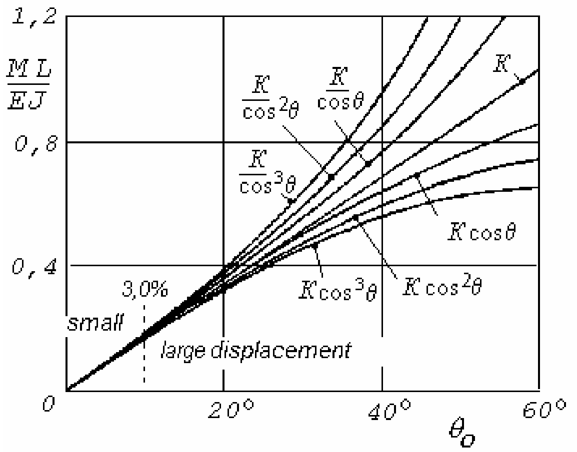

Fig. 3. *Connection of stress and angular deflection of bar cross-section at different curvature expressions*

5. Parametric form of curvature expression

Apart from physical meaning of elastics problems that the Y. Bernoulli equation (1) imparts them, it is a purely mathematical problem – «line reestablishment» by the function of curvature change $f(k)=(M/EJ)$. Mathematics gives the definition of curvature (2) and its coordinate expression (3), the assignment of which is to state the curvature of functionally determined line. The formulas (2) and (3) are not intended for solution of inverse problems, such problems are not considered in the course of mathematics.

By the expressions (2) and (3) in parametric form [10]

$$K = \mp\frac{d(\sin\theta)}{dx} = \pm\frac{d(\cos\theta)}{dy} \tag{7}$$

the curvature is determined in the system of coordinates by the rate of goniometric function change in tangent inclination angle θ in a point when moving along the line. Here the tangent inclination angle is presented as a basic parameter. The simplified expression acquire similar form (4)

$$K^* = \pm\frac{d(\mathrm{tg}\theta)}{dx} = \pm\frac{d}{dy}\left(\frac{\mathrm{tg}^2\theta}{2}\right). \tag{8}$$

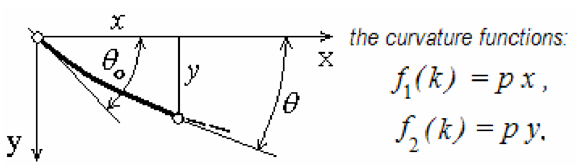

Fig. 4. *Line reestablishment by curvature*

Curvature in these expressions has the signs, they being determined by the change in corresponding functions that are characterised by line curvature. Structure of exact and simplified curvature expressions visualizes connections and community of the two theories.

So, with the curvature function $f_1(k)=px$ (Fig. 4), with the curvature expressions (7) and (8) the problem of line reestablishment is formulated by the differential equations

$$-\frac{d(\sin\theta)}{dx}=px,\quad +\frac{d(\cos\theta)}{dy}=px. \qquad (9)$$

$$-\frac{d(\mathrm{tg}\theta)}{dx}=px,\quad -\frac{d(\mathrm{tg}^2\theta)}{2dy}=px. \qquad (10)$$

Under initial conditions $x=0$, $y=0$, $\theta=\theta_0$ from (9) the equation of line for any curvatures follows:

$$x\sqrt{p}=\sqrt{2(\sin\theta_0-\sin\theta)},$$

$$y\sqrt{p}=\int_{\theta_0}^{\theta}\frac{d(\cos\theta)}{\sqrt{2(\sin\theta_0-\sin\theta)}}. \qquad (11)$$

For «small» deformations from (10) its equation:

$$x\sqrt{p}=\sqrt{2(\mathrm{tg}\theta_0-\mathrm{tg}\theta)},$$

$$y\sqrt{p}=\frac{1}{3}(2\mathrm{tg}\theta_0+\mathrm{tg}\theta)\sqrt{2(\mathrm{tg}\theta_0-\mathrm{tg}\theta)}. \qquad (12)$$

Similarly, with the curvature function $f_2(k)=py$ (Fig. 4) we state the equations of other curve:

$$y\sqrt{p}=\sqrt{2(\cos\theta-\cos\theta_0)},$$

$$x\sqrt{p}=\int_{\theta_0}^{\theta}\frac{d(\sin\theta)}{\sqrt{2(\cos\theta-\cos\theta_0)}}. \qquad (13)$$

$$y\sqrt{p}=\sqrt{\mathrm{tg}^2\theta_0-\mathrm{tg}^2\theta},\ x\sqrt{p}=\frac{\pi}{2}-\arcsin\left(\frac{\mathrm{tg}\theta}{\mathrm{tg}\theta_0}\right). \qquad (14)$$

It should be noted, in terms of curvature functions and initial conditions, it is possible to define only the equation of line. Elastics problems are boundary-value ones. The range of point coordinates for the curved line of finite length is limited by its projection of the coordinate axis and it remains undefined. It is to be determined by solving the problem of defining arc length at the indicated position of finite point. Such a problem is called a line «flattering» in mathematics. At its «large» and «small» deformations it is formulated in the same way:

$$L=\int_{\theta_0}^{\theta}\frac{dx}{\cos\theta}\ \text{or}\ L=\int_{\theta_0}^{\theta}\frac{dy}{\sin\theta}.$$

In the theory of «small» displacements it is not solved, indefiniteness is removed by the assumption $dx=dL$, which is equivalent to $L=x_L$. L.I. Sedov [7] determined this assumption as «linearization of boundary conditions».

Fig. 5 presents «elastics» in terms of the equations (11, 13) and «deflection curves» in terms of (12, 14) of the bar with *EJ* hardness at curvature by concentrated stress *P*. Hence, in the curve equations $p=P/EJ$. Different kinds of deformation are formed by pointing out the position of end cross-section.

Determining deformation geometry of bars presentation of two problems («reestablishment» and «flattering» of line) as one sometimes makes difficult to interpret the solution and can result in incorrect conclusions. Let us show this by the solution of «Euler's problem» (the bar with two joint as supports in Fig. 5).

By the expressions (14) periodic curve is determined. Having pointed out the position of end cross-section $x=x_L$, $\theta=-\theta_0$, from them we obtain $y=0$, $x_L\sqrt{p}=\pi$. Reducing them to the form

$$y=\frac{\mathrm{tg}\theta_0}{\sqrt{p}}\sqrt{1-\left(\frac{\mathrm{tg}\theta}{\mathrm{tg}\theta_0}\right)^2},\quad \frac{\mathrm{tg}\theta}{\mathrm{tg}\theta_0}=\cos(x\sqrt{p}),$$

we define it by the equation in coordinate form:

$$y=\frac{\mathrm{tg}\theta_0}{\sqrt{p}}\sin(x\sqrt{p}). \qquad (15)$$

The curve is sinusoid with the amplitude

$$f=y_{\max}=\frac{\mathrm{tg}\theta_0}{\sqrt{p}}=\frac{x_L}{\pi}\mathrm{tg}\theta_0.$$

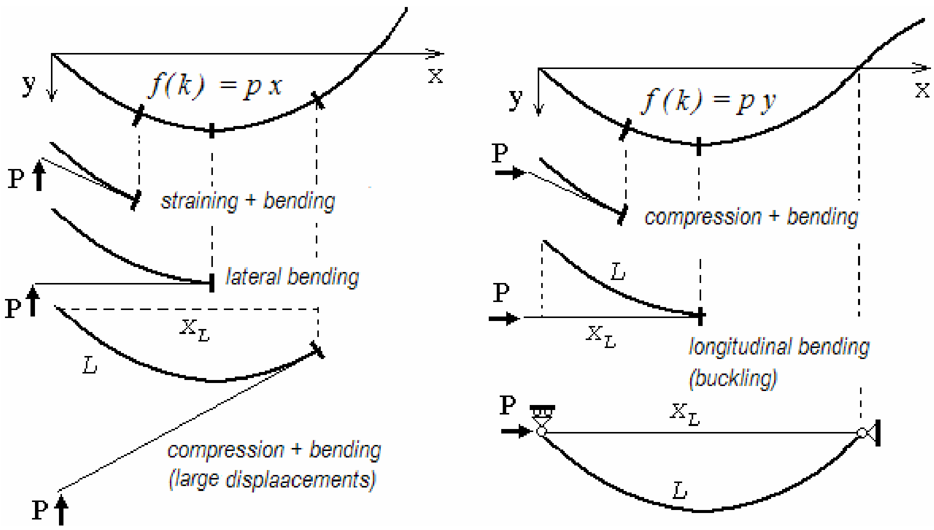

Fig. 5. *Kinds of deformation determined by one curve*

At «small» deflection» in the assumption $L \approx x_L$ one half-wave is isolated from it.

Solution of the L. Euler's problem has formed the concept on existence of a number of buckling loads [5]. The equation (15) with indefinite deflection in its solution assumes interpretation of $\sin(x\sqrt{p})=0$ as $x\sqrt{p}=n\pi$, and coefficient p of the curvature function is takes as a cause for multiplicity.

Conclusions

In elastics problems of curved bars the functions of curvature change formulated by stresses are usually coordinate ones. The questions of agreement of these functions with curvature expressions conditioned appearance of two theories. Its agreement with the curvature expression (2) leads to the theory of «large» displacements with corresponding specific character of problem solution through «Kirchhoff's dynamic analogue». Its agreement with the expression (3) simplified in terms of (5) for «linearization of differential equations» and assumption $dL=dx$ for «linearization of boundary conditions» results in the theory of «slight» displacements with «rules and principles», the main point of which is expressed by the performed assumptions. Their indistinct interpretation formed a supposedly strict difference between the theories. Parametric curvature expressions (exact and simplified ones) in coordinate form remove the questions of agreement with its function of change and show conditionality and irrationality of the concept of fundamental difference of the two theories.

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