How to transform all multiple solutions of the Kemeny Ranking Problem into a single solution

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Abstract. Preference aggregation as a problem of a single consensus ranking determination, using Kemeny rule, for *m* rankings, including ties, of *n* alternatives is considered in the paper. The Kemeny Ranking Problem (KRP) may have considerably more than one optimal solutions (strict orders or permutations of the alternatives) and, hence, special efforts to deal with this phenomenon are needed. In the paper, there is proposed an efficient formal rule for convolution of the *N* multiple optimal permutations, the output profile B(N, n), into an exact single final consensus ranking, which can include ties. The convolution rule is as follows: in the final consensus ranking, alternatives are arranged in ascending order of their rank sums (total ranks) calculated for the output profile B; some two alternatives are considered to be tolerant if they have the same rank sums in B. The equivalent convolution rule can be also applied as follows: in the final consensus ranking, alternatives are arranged in descending order of row sums (total scores) calculated for a tournament table built for B; some two alternatives are deemed to be tolerant if they have the same row sums. It is shown that, for any alternative, its total rank and total score are equal in sum to the output profile dimension $N \times n$. The convolution rules are validated using Borda count.

1. Introduction

In this paper we consider a preference aggregation procedure in the form of Kemeny rule. Being essentially the multiple heterogeneous properties measurement in an ordinal scale, preference aggregation belongs to a circle of problems relevant to measurement science [1]. Additionally, in the recent few years, the preference aggregation and the processing of ranked data have attracted considerable interest within a much broader consideration covered by such a rapidly developing area of artificial intelligence as *machine learning* [2]. Appropriate methods are used to solve newly emerging problems, such as crowd sensing/labeling, sentiment analysis, meta-search engines, interval data fusion and others [2–7].

The preference aggregation is a single consensus ranking determination for *m* rankings (voters), possibly including ties, of *n* alternatives (candidates). This is a classical problem that has been intensively investigated firstly as a Voting Problem in the framework of Social Choice Theory [8].

Let we have a set $A = \{a_1, a_2, ..., a_n\}$ of *n* alternatives and a set $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ of *rankings* defined over the set *A*, where each of *m* rankings (also called *preference relation* or *weak order*) has a form of chain of size n, e.g. $\lambda = a_2 \succ a_1 \succ ... \succ a_s \sim a_t \succ ... \succ a_p \sim a_q$, and may include $\succ$, a *strict preference* relation ρ, and $\sim$, a *tolerance* relation (or *tie*) τ, so that $\lambda = \rho \cup \tau$. The set $\Lambda(m, n)$ is called a *preference profile* for the given m rankings of n alternatives.

We will use the following obvious *short notation* for a ranking:

$$\lambda_k = a_2 \succ a_1 \succ ... \succ a_s \sim a_t \succ ... \succ a_p \sim a_q \Leftrightarrow 21\ldots(st)\ldots(pq). \quad (1)$$

For example, $a_2 \succ a_5 \sim a_1 \succ a_3 \sim a_4 \succ a_6 \Leftrightarrow 2(51)(34)6$ or $a_3 \succ a_1 \succ a_5 \succ a_2 \succ a_6 \succ a_4 \Leftrightarrow 315264$.

Let Π_n be a set of all $n!$ strict order relations $\succ$ on A; each strict order $\rho \in \Pi_n$ is in one-to-one correspondence with a *permutation* of first n natural numbers $\infty_n = \{1, 2, \ldots, n\}$; in this sense, we will treat notions of strict (or linear) order and permutation as synonyms.

Preference aggregation goal is to determine a *consensus ranking* β that is a single preference relation that would give an integrative characterization of the profile $\Lambda(m, n)$.

There are many rules to find the consensus ranking [9]. We use the *Kemeny rule* [10] consisting in determination of such linear order (*Kemeny ranking*) $\beta \in \Pi_n$ of the alternatives of A that the distance $D(\beta, \Lambda)$ (defined in terms of the *number of pairwise inconsistencies* between rankings) from β to the rankings of the initial profile $\Lambda(m, n)$ is minimal for all possible strict orders ρ (permutations), that is

$$\beta = \arg\min \sum_{i<j} p_{ij} , \quad (2)$$

where $p_{ij} = \sum_{k=1}^{m} [1 - \mathrm{sgn}(a_i^k, a_j^k)],\ i, j = 1,...,n,$ is an element of the $(n \times n)$ *profile matrix* $[p_{ij}] = P$, rows and columns of which are labeled by the alternatives' numbers (indexes); $\mathrm{sgn}(a_i, a_j) = \begin{cases} 1, & a_i \succ a_j \\ 0, & a_i \sim a_j \\ -1, & a_i \prec a_j \end{cases}$ is a function that reveals the sign (or direction) of the pair $(a_i, a_j) \in \lambda$.

The Kemeny rule (2) is an optimization problem, called the *Kemeny Ranking Problem (KRP)*, which means the determination of such a transposition of the profile matrix rows and columns that the sum of elements of its upper triangle submatrix, i.e. $D(\rho, \Lambda) = \sum_{i<j} p_{ij}$, is minimal.

Clearly since the set Π_n is finite, the Kemeny ranking always exists, however it is *not always unique* as the optimal distance value may correspond to not the only transposition of the profile matrix rows and columns, that is $D(\beta_1, \Lambda) = \sum_{i<j} p_{ij}^1 = D(\beta_2, \Lambda) = \sum_{i<j} p_{ij}^2 = ... = D(\beta_N, \Lambda) = \sum_{i<j} p_{ij}^N$, where N is a number of Kemeny rankings (optimal permutations) [2,3,11,12]. Furthermore, it was shown in [1] that in certain situations the number of multiple optimal permutations may exceed 10^7 even for small $m = 4$ and $n = 15$ and also the solutions may rank the alternatives in significantly different ways what produces much more ambiguity than that inherent in the initial profile Λ.

Example 1. Let the following profile be given: $\Lambda(3,3) = \{1{\sim}32;\ 31{\sim}2;\ 2{\sim}31\}$. Then the KRP has two solutions: $\beta_1 = 312$ and $\beta_2 = 321$, both having minimal distance to the profile $D(\beta_1, \Lambda) = D(\beta_2, \Lambda) = 5$. The considered profile $\Lambda(3,3)$ is shown in figure 1 as belonging to the weak order space for $n = 3$. □

In order to solve the KRP we use the recursive algorithm of our own design, RECURSALL, implementing the branch & bound technique, allowing to find *all possible* Kemeny rankings for a given preference profile [4]. It is convenient to term the initial preference profile $\Lambda(m, n)$ as *input profile* and the multiple optimal permutations as output profile $\mathrm{B}(N, n) = \{\beta_1, \beta_2, ..., \beta_N\}$, $\mathrm{B} \subset \Pi_n$ (see figure 2).

Clear that special efforts to deal with the phenomenon of the KRP multiple optimal solutions, where $N_s >> m$, are needed. The rest of the paper is devoted to development of an efficient formal rule for convolution of the output profile into a single final consensus ranking β_{fin}.

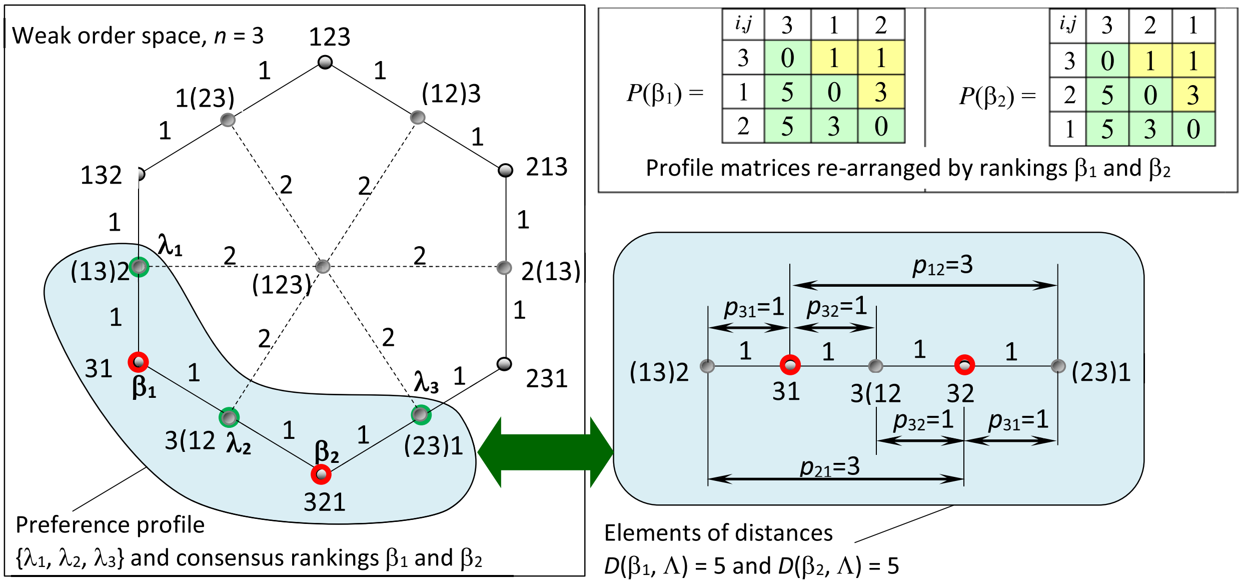

Figure 1. Toward s multiplicity of the KRP solutions: preference profile $\{\lambda_1, \lambda_2, \lambda_3\}$ and consensus rankings β_1 and β_2 in the weak order space for $n = 3$.

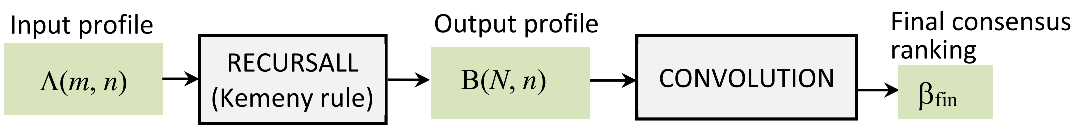

Figure 2. Stages of the final consensus ranking determination.

2. Convolution rule for the KRP multiple solutions

A rule for convolution of the set of optimal permutations can be formulated as follows. Let $B(N, n) = \{\beta_1, \beta_2, \ldots, \beta_N\}$, $B \subset \Pi_n$, be an output profile of the KRP applied to an input profile $\Lambda(m, n) = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ given over some set of alternatives $A = \{a_1, a_2, ..., a_n\}$ and *rank* r_i^k be a position of an alternative a_i in the consensus ranking $\beta_k \in B$, $k = 1, \ldots, N$. Let a *total rank* r_i of the alternative a_i is defined as $r_i = \sum_{k=1}^{N} r_i^k$. Then, for all $i < j$, $i, j = 1, \ldots, n$,

$$r_i < r_j \Rightarrow a_i \succ a_j \text{ and } r_i = r_j \Rightarrow a_i \sim a_j, \tag{3}$$

where both relations $\succ$ and $\sim$ are in the single final consensus ranking β_{fin}.

Thus, the final consensus ranking may include tolerances as well as an arbitrary $\lambda_k \in \Lambda$, i.e. generally speaking, $\beta_{fin} = \rho \cup \tau$ and $\beta_{fin} \notin \Pi_n$.

Example 2. Table 1 and figure 3 show an example of convoluting two optimal permutations into a single β_{fin}. One can see that alternatives 1, 3, 5, and 6 have different positions in two optimal solutions β_1 and β_2,

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however, their distances to the initial profile are $D(\beta_1, \Lambda) = D(\beta_2, \Lambda) = 58$. The convolution rule (3) allows to find their true positions in β_{fin}, namely as in the ranking 421(35)6. □

Table 1. Example of the convolution rule (3) application.

Input profile Λ	Output profile B	Ranks of alternatives in B							Final solution β_{fin}
λ_1: 1 2 6 4(3 5)			a_1	a_2	a_3	a_4	a_5	a_6	Short form:
λ_2: 4 5 1 2 3 6	β_1: 4 2 1 3 6 5	r_i^1	3	2	4	1	6	5	4 2 1 (35) 6
λ_3: 2(51)(34)6	β_2: 4 2 5 1 3 6	r_i^2	4	2	5	1	3	6	Extended form:
λ_4: (63)4 2(15)		r_i^1 □ r_i^2	**7**	**4**	**9**	**2**	**9**	**11**	$a_4 \succ a_2 \succ a_1 \succ a_3 \sim a_5 \succ a_6$
λ_5: 3 4 (26) 51									

The convolution rule (3) guarantees determination of an exact single optimal permutation β_{fin} for any output profile B.

	1	2	3	4	5	6
1	0	6	4	6	6	4
2	4	0	4	6	2	3
3	6	6	0	5	5	3
4	4	4	5	0	2	4
5	4	8	5	8	0	6
6	6	7	7	6	4	0

Initial profile matrix *P*

	4	2	1	3	6	5
4	0	4	4	5	4	2
2	6	0	4	4	3	2
1	5	6	0	4	4	6
3	5	6	6	0	3	5
6	6	7	6	7	0	4
5	8	8	4	5	6	0

The profile matrix re-arranged by consensus ranking β_1

	4	2	5	1	3	6
4	0	4	2	4	5	4
2	6	0	2	4	4	3
5	8	8	0	4	5	6
1	6	6	6	0	4	4
3	5	6	5	6	0	3
6	6	7	4	6	7	0

The profile matrix re-arranged by consensus ranking β_2

Figure 3. Profile matrices from Example 2.

3. Convolution rule justification

Let the output profile B(N, n) be represented by the ($n\times n$) *tournament table* $S = [s_{ij}]$, where $s_{ij} = \sum_{k=1}^{N} \mathbf{I}_\rho(a_i^k, a_j^k), i, j = 1,...,n$, is the matrix element and $\mathbf{I}_\rho(a_i, a_j) = \begin{cases} 1 \text{ if } (a_i, a_j) \in \rho \\ 0 \text{ if } (a_i, a_j) \notin \rho \end{cases}$ is the indicator function[1].

A sum of elements of i-th row of the table S is termed as the *score* z_i of the alternative a_i, i.e. for $i = 1, \ldots, n$, $z_i = \sum_{j=1}^{n} s_{ij}$. Then multiple permutations can be also convoluted using the scores z_i as follows: for all $i < j$, $i, j = 1, \ldots, n$,

$$z_i > z_j \Rightarrow a_i \succ a_j \text{ and } z_i = z_j \Rightarrow a_i \sim a_j, \tag{4}$$

where both relations $\succ$ and $\sim$ are in the single final consensus ranking β_{fin}.

The convolution rule (4) guarantees determination of an exact single optimal permutation β_{fin} for any output profile B, and the single optimal permutation is the same as that found using the rule (3). In fact, the rule (4) coincides with the preference aggregation rule well known as Borda count [13].

[1] For the indicator function, the following statements are hold: $(a_i, a_j) \in \rho \Leftrightarrow a_i \succ a_j$ and $(a_i, a_j) \notin \rho \Leftrightarrow a_i \prec a_j$.

For any output profile B(N, n) represented by a corresponding tournament table [s_{ij}], the following equation is valid:

$$r_i = N \cdot n - z_i,\ i = 1,...,n\ . \tag{5}$$

Example 3. Table 2 shows the values of scores and rank sums calculated for the profile alternatives from Example 2. The alternatives are sorted in accordance with rules (3) and (4). It can be seen the fairness of expression (5) for these data.

Table 2. Scores and rank sums of the output profile B from Example 2.

Alternatives	a_4	a_2	a_1	a_3	a_5	a_6
Total score z_i	10	8	5	3	3	1
Total rank r_i	2	4	7	9	9	11
$N{\cdot}n = z_i + r_i$	12	12	12	12	12	12

The expression (5) can serve as a basis for justification of the convolution rule (3) validity. Indeed, let a *social score function f* assigns a nonempty choice set $C \subset A$ to a profile B. It was proved by H.P. Young in [13] that "for any fixed number n of alternatives, there is one and only one social score function that is neutral, consistent, faithful, and has cancellation property – namely, Borda's rule". The properties of the function f listed in the Young's statement are deemed as the set of axioms necessary and sufficient in order to have a fully characterized preference aggregation rule [14]. Thus, the scores z_i, implementing Borda rule, guarantee that the rule (4) leads to convolution of the output profile B(N, n) into the exact and single consensus ranking β_{fin}. It follows from (5) that social score functions f coincide for the rules (3) and (4).

4. Conclusion

It was shown in the paper that the multiple optimal permutations, i.e. the output profile B(N, n) = {β_1, β_2, ..., β_N}, as solutions of the KRP for m rankings, including ties, of n alternatives can be efficiently transformed into an exact single final consensus ranking β_{fin}, which can also include ties. The proposed convolution rule is as follows: in the final consensus ranking β_{fin}, alternatives are arranged in ascending order of their rank sums calculated for the output profile B; in β_{fin} some two alternatives are tolerant to each other if they have the same rank sums in the output profile. The equivalent convolution rule can be also applied as follows: in the final consensus ranking β_{fin}, alternatives are arranged in descending order of row sums (total scores) z_i calculated for a tournament table S of the output profile B; in β_{fin} some two alternatives are tolerant to each other if they have the same row sums of the tournament table.

It was also shown that, for any i-th alternative, its total rank r_i and total score z_i are equal in sum to a constant value $N{\cdot}n$, that is the dimension of the output profile B. The convolution rules validity has been shown using such well known preference aggregation rule as Borda count.

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