On automatic tuning of basis functions in Bezier method

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Abstract. A transition from the fixed basis in Bezier's method to some class of base functions is proposed. A parameter vector of a basis function is introduced as additional information. This achieves a more universal form of presentation and analytical description of geometric objects as compared to the non-uniform rational B-splines (NURBS). This enables control of basis function parameters including control points, their weights and node vectors. This approach can be useful at the final stage of constructing and especially local modification of compound curves and surfaces with required differential and shape properties; it also simplifies solution of geometric problems. In particular, a simple elimination of discontinuities along local spline curves due to automatic tuning of basis functions is demonstrated.

1. Formulation of the problem

Let us have segment C of some curve. We represent this segment in Bezier form [1–6] by radius-vectors r_i and weights $w_i \, (w_i > 0)$ of the four vertices of characteristic polyline L. This segment has the following analytical description:

$$\mathbf{w}(t)\mathbf{r}(t) = \sum_{i=0}^{3} w_i r_i f_i(t), \; \mathbf{w}(t) = \sum_{i=0}^{3} w_i f_i(t), \tag{1}$$

where t is a parameter ranging from t_1 to $t_2 \; (t_2 > t_1)$;

$\mathbf{r}(t)$ and $\mathbf{w}(t)$ – the radius-vector and the weight of a current point on segment C;

$f_i(t)$ – basis functions.

We will consider curves in a 3-dimensional Euclidean space; we will also consider the so called "curves on the surface", presented at the plane of parameters u, v, which define this surface by equation $P = P(u, v)$ [1, 3, 4, 6]. Then we have $\mathbf{r}(t) = (x(t), y(t), z(t))$, $\mathbf{r}_i = (x_i, y_i, z_i)$ in the first case and $\mathbf{r}(t) = (u(t), v(t))$, $\mathbf{r}_i = (u_i, v_i)$ in the second case. Radius vectors $\mathbf{r}_i$ are called control points [6]. The two equations above (1) can be replaced with one by introducing homogeneous coordinates [1–6].

We will consider the case with the minimum number of vertices of the polyline. Then for any two non-planar segments C_1 and C_2, there exists an affine or projective transformation from C_1 to C_2. These transformations are reduced to the transformation of vertices and their characteristic polyline [3, 6]. Such transformations simplify the solution of geometric problems. We can extend the class of transformations [3, 6] including the birational transformations, a special case of which is the inversion that has a property to transform a circle into a straight and vice versa.

Birational transformations can be used [7] to develop effective algorithms for such difficult tasks as construction of the line of intersection of two surfaces in solid modeling of geometric objects. The limitation to the number of vertices on polyline L may be compensated in part by extending function class fi. Let us consider the following basis functions:

$$f_0 = a_0\lambda\mu^2 + \mu^3, \ f_1 = a_1\lambda\mu^2, \ f_2 = a_2\lambda^2\mu,$$
$$f_3 = a_3\lambda^2\mu + \lambda^3 \ (a_0 \geq 0, \ a_1 > 0, \ a_2 > 0, \ a_3 \geq 0),$$

(2)

where a_i are constants, and $\lambda, \ \mu$ – linear functions of parameter t :

$$\lambda = \frac{b_1(t-t_1)}{t_2-t_1}, \ \mu = \frac{b_2(t_2-t)}{t_2-t_1}, \ (0 < b_1 \leq 1, \ 0 < b_2 \leq 1).$$

(3)

Comment 1. Any basis that belongs to the class considered in this paper can be transformed into the Bernstein basis by changing the characteristic polyline L in the following manner:

$$\mathbf{r}_0^* = \mathbf{r}_0, \ \mathbf{r}_1^* = \frac{a_0w_0\mathbf{r}_0 + a_1w_1\mathbf{r}_1}{a_0w_0 + a_1w_1}, \ \mathbf{r}_2^* = \frac{a_2w_2\mathbf{r}_2 + a_3w_3\mathbf{r}_3}{a_2w_2 + a_3w_3}, \ \mathbf{r}_3^* = \mathbf{r}_3,$$

$$w_0^* = b_2^3w_0, \ w_1^* = \frac{b_1b_2^{\ 2}(a_0w_0 + a_1w_1)}{3}, \ w_2^* = \frac{b_1b_2^{\ 2}(a_0w_0 + a_1w_1)}{3}, \ w_3^* = b_1^3w_3.$$

Thus, the extension of the two-parameter class of basis functions considered in [2, 3] is proposed. Let us introduce vector $\mathbf{S} = (t_1, \ t_2, \ b_1, \ b_2, \ a_0, \ a_1, \ a_2, \ a_3)$ for parameters of functions f_i. Here t_1 and t_2 are nodes, b_1 and b_2 are their weights on the parametric line. In special cases, we have the Bernstein basis at $\mathbf{S} = (t_1, \ t_2, \ 1, \ 1, \ 0, \ 3, \ 3, \ 0)$ and the Ball basis at $\mathbf{S} = (t_1, \ t_2, \ 1, \ 1, \ 1, \ 2, \ 2, \ 1)$ [1–6].

So, let characteristic polyline L (with vertexes $\mathbf{r}_0, \ \mathbf{r}_1, \ \mathbf{r}_2, \ \mathbf{r}_3$) and vector $\mathbf{S}$ are defined with regard to the specified limitations. Then rational cubic segment C that tangent the control polyline in its end points $\mathbf{r}_0, \ \mathbf{r}_3$ and belongs to its convex hull is uniquely determined [1, 3]. A rational quadratic segment (segment of a conic section) is a special case here and has a canonical definition [3] under the following conditions:

$$\mathbf{r}_1 = \mathbf{r}_2, \ w_0 = w_3 = 1, \ w_1 = w_2, \ \mathbf{S} = (0, \ 1, \ 1, \ 1, \ 1, \ 2, \ 2, \ 1).$$

(4)

The conic segment can always be converted into a canonical form by an appropriate parametrization of the curve [3].

The change of $t_1, \ t_2, \ b_1, \ b_2$ means a transfer to a new parametrization of the curve that preserves its shape [1, 3]. At this, the length of tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ changes in the end points of segment C. The law of point distribution along the curve also changes. Here, we have a risk [3, 6] of significantly non-uniform distribution of points along the segment of the curve, e.g. on the interval of the straight or on the arc of the circle at equidistant values of parameter t . When values of a_0, a_1, a_2, a_3 are changed, the shape of the curve is usually modified; in particular curvatures K_1 and K_1 are changed in the end points of the curve segment.

By controlling the parameters of the basis functions (vector $\mathbf{S}$), it is possible to refine the originally built segment of the curve, to obtain the segments of different properties, which are useful in the construction of the local spline [5] close to the physical [8] one on the accuracy of the approximation, the differential properties (smoothness, curvature continuity) and property forms (with the exception of the oscillations, the monotonous change of the curvature between a spline nodes).

This approach is useful for
• the local approximation, interpolation and smoothing of the curve given by discrete points [5, 6, 9–11];
• the construction and local modification of a generalized cylinder [3] or a generalized cone [12] with required differential properties and properties of the shape its side surface;
• in the construction line of intersection of two surfaces [4, 7];

- solving the problem complex conjugate surfaces and bodies [6, 13].

Let us consider two problems that are important from this perspective.

Problem 1. Given: characteristic polyline L and vector $\mathbf{S}$ defining the conical segment in the canonical form (4), which is not a line segment; some curvatures K_1^* and K_2^*.

Required: to modify segment C by tuning vector $\mathbf{S}$, so that resulting segment C^* has curvatures K_1^* and K_2^* at its end points, and the tangent vectors (at these points) have kept their length $\left(\left| \mathbf{T}_1^* \right| = \left| \mathbf{T}_1 \right|, \; \left| \mathbf{T}_2^* \right| = \left| \mathbf{T}_2 \right| \right)$.

Comment 2. Conservation of the length of tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ in transformation of a curve segment is important in solving such problems as local modification of a compound surface when its smoothness should remain [6] and in construction of the line of intersection of two surfaces with a posteriori estimated accuracy, when isoparametric [4, 14] or birational [7] correspondence between points on the intersection line is represented in a 3-dimensional Euclidean space and in the parameters space of such surfaces.

Problem 2. This task is to evaluate the deviation of modified segment C^* obtained by solving problem 1, from initial segment C.

It is known [1] that an attempt to secure some preset curvature values in the end points of a cubic Bézier segment (at $w_i = 1$) by modifying its control poliline should result in a fourth-degree equation that may have no real roots.

An algorithmic solution of this problem for a rational cubic segment in the Ball basis is proposed in [3] by means of a simultaneous change of location of control points $\mathbf{r}_1$, $\mathbf{r}_2$ and weights w_1, w_2 when $K_1^* \neq 0$ and $K_2^* \neq 0$. Ensuring continuity of curvature of local spline obtained in [10] at the cost of using two cubic segments between each pair of spline nodes (similar to biarcs composed of two circular arcs and used for splines piecewise constant curvature [5, 11, 15], and in case of paired bodies [13]). Curvature persistence of a local spline was obtained in [10] by using two cubic segments between each pair of spline nodes (similar to biarcs composed of two circular arcs and used in splines of piecewise constant curvature [5, 11, 15] and in conjunction of bodies [13]).

In this paper, we demonstrate the possibility of a simpler and more efficient solution of the same problem by means of an automatic tuning of basis functions for each segment of a local spline.

2. Local modification of curves

Let rational cubic segment C be represented by characteristic polyline L and vector $\mathbf{S}$. Let rational cubic segment C be represented by characteristic polyline L and vector $\mathbf{S}$. From the analytical description of this spline (1–3), it follows that:

$$\mathbf{r}(t_1) = \mathbf{r}_0, \; w(t_1) = b_2^3 w_0, \; \mathbf{r}(t_2) = \mathbf{r}_3, \; w(t_2) = b_1^3 w_3. \tag{5}$$

Differentiating equation (1) with respect to t and then substituting expression $w' = dw/dt$ from the second equation into the first, we obtain an equation for tangent vector $\mathbf{T}$ at arbitrary point C:

$$\mathbf{T} = \mathbf{r}' = \sum_{i=0}^{3} w_i (\mathbf{r}_i - \mathbf{r}) f_i' / w.$$

From this and on the basis of (2), (3) and (5) we have:

$$\mathbf{T}_1 = \mathbf{r}'(t_1) = \frac{a_1 b_1 w_1 (\mathbf{r}_1 - \mathbf{r}_0)}{b_2 w_0 (t_2 - t_1)}, \; \mathbf{T}_2 = \mathbf{r}'(t_2) = \frac{a_2 b_2 w_2 (\mathbf{r}_3 - \mathbf{r}_2)}{b_1 w_3 (t_2 - t_1)}. \tag{6}$$

Differentiating twice equation (1) and eliminating w'', we obtain the following equation:

$$w \mathbf{r}'' = \sum_{i=0}^{3} w_i (\mathbf{r}_i - \mathbf{r}) f_i'' - 2 w' \mathbf{r}'.$$

At the next step, we obtain the equation that defines curvature $K = K(t)$ and unit binormal vector $\mathbf{B} = \mathbf{B}(t)$ at arbitrary point on segment C by using the well-known equation from differential geometry:

$$K \cdot \mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}'|^3} = \frac{\mathbf{T} \times \sum_{i=0}^{3} w_i (\mathbf{r}_i - \mathbf{r}) f_i''}{w |\mathbf{T}|^3}.$$

Thus, we obtain the following equation for the end points of segment C, taking into account (2), (3), (5) and (6):

$$K_1 \cdot \mathbf{B}_1 = \frac{2w_0}{a_1^2 w_1^2} \cdot \frac{(\mathbf{r}_1 - \mathbf{r}_0) \times [a_2 w_2 (\mathbf{r}_2 - \mathbf{r}_1) + a_3 w_3 (\mathbf{r}_3 - \mathbf{r}_{0)})]}{|\mathbf{r}_1 - \mathbf{r}_0|^3},$$

$$\tag{7}$$

$$K_1 \cdot \mathbf{B}_2 = \frac{2w_3}{a_2^2 w_2^2} \cdot \frac{[a_1 w_1 (\mathbf{r}_2 - \mathbf{r}_1) + a_0 w_0 (\mathbf{r}_3 - \mathbf{r}_{0)})] \times (\mathbf{r}_3 - \mathbf{r}_2)}{|\mathbf{r}_3 - \mathbf{r}_2|^3}.$$

When $a_0 = a_3 = 0$, $a_1 = a_2 = 3$, equations (7) coincide with equations [1] of the Bernstein basis and when $a_0 = a_3 = 1$, $a_1 = a_2 = 2$ – with the equations [2, 3] of the Ball basis.

It follows from equations (6) that tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ do not depend on parameters a_0 and a_3 of basis functions f_0 and f_3. It should be additionally noted that parameter a_0 is linearly included only in the second equation and parameter a_3 – only in the first equation in (7).

Let us now consider Problem 1. Since we will need to save polyline L and fulfil conditions $\mathbf{T}_1^* = \mathbf{T}_1$, $\mathbf{T}_2^* = \mathbf{T}_2$, it is reasonable to modify segment C by means of tuning parameters a_0 and a_3. Thus, simple formulas are derived from equations (7) and conditions (4):

$$a_0^* = K_2^* / K_2, \quad a_3^* = K_1^* / K_1. \tag{8}$$

Here, a solution always exists because conic segment C is not an interval of line $(K_1 > 0, K_2 > 0)$ as defined in Problem 1. If $K_1^* = K_2^* = 0$, then the modification will produce rational cubic segment C^* of zero curvature values in its end points.

Therefore, a simple solution can be obtained for the problem of conjunction of segments in a compound curve when curvature persistence should be provided (including the cases when some segments are line intervals). It is clear that eliminating the curvature discontinuities of a smooth composite curve in this manner, we can always ensure the following conditions:

$$w_1 \neq w_2 \quad 0 \leq a_0^* \leq 1, \ 0 \leq a_3^* \leq 1 \ \left(0 \leq K_1^* \leq K_1, \ 0 \leq K_2^* \leq K_2 \right). \tag{9}$$

Notice, that formulas (8) are fair for a wider class of source segments C compared to conic segments because we only use condition $\mathbf{r}_1 = \mathbf{r}_2$ of conditions (4). In particular, segment C may be a rational cubic segment at $w_1 \neq w_2$. Arbitrary parameterization of the curve is also possible when $b_1 \neq b_2$.

Example. Some segment C is represented by its control points $\mathbf{r}_0 = (0, 1, 0)$, $\mathbf{r}_1 = \mathbf{r}_2 = (0, 0, 0)$ and $\mathbf{r}_3 = (1, 0, 0)$, weights $w_0 = w_3 = 1$, $w_1 = w_2 = \sqrt{2} / 2$ and vector $\mathbf{S} = (0, 1, 1, 1, 1, 2, 2, 1)$. This is an arc of unit circle $(K_1 = K_2 = 1)$ [3]. Let segment C^* should have curvature values $K_1^* = 0$ and $K_2^* = 1$ in end points $\mathbf{r}_0$ and $\mathbf{r}_3$. Then $a_0^* = 1$ and $a_3^* = 0$ according to formulas (8).

Resulting segment C^* together with the initial segment of C is shown in figure 1, which also shows the isoparametric correspondence between the points of segments, calculated at a constant step of parameter t, equal to 0.1. Let us note that in this case, we could establish an isoparametric

correspondence close to a birational one ("almost on a normal" to segment C) by tuning parameters b_1 and b_2, i.e. changing parametrization of segments C and C^*.

3. A posteriori estimation of accuracy

Let us turn to problem 2 and consider only the case when conditions (4) and (9) are fulfilled simultaneously. Let us introduce vector $\mathbf{h} = \mathbf{r}_1 - \mathbf{r}_1^*$, where $\mathbf{r}_1^*$ is a projection of control point $\mathbf{r}_1$ on the line that passes through points $\mathbf{r}_0$ and $\mathbf{r}_3$. Its length $h = |\mathbf{h}|$ is the height of the control triangle with vertexes $\mathbf{r}_0$, $\mathbf{r}_1 = \mathbf{r}_2$ and $\mathbf{r}_3$ (see figure 1) and single vector $\mathbf{e} = \mathbf{h}/h$ is perpendicular to vector $\mathbf{r}_3 - \mathbf{r}_0$.
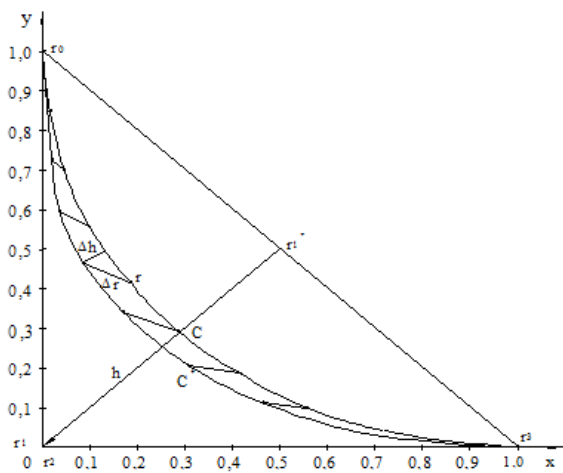

Figure 1. Isoparametric correspondence between points of the segments.

Let $\Delta\mathbf{r}(t)$ be the vector defining isoparametrical deviation of current point $\mathbf{r}^*$ on segment C^* from corresponding point $\mathbf{r}$ on conic segment C, i.e. $\Delta\mathbf{r}(t) = \mathbf{r}^*(t) - \mathbf{r}(t)$. Let us introduce function $\Delta h = \mathbf{e} \cdot \Delta\mathbf{r}$, which is a scalar projection of vector $\Delta\mathbf{r}$ on the direction of vector $\mathbf{h}$.

Let us call a deviation of segment C^* from segment C a maximum of function $|\Delta h|$ on interval [0, 1], where $\Delta h = \Delta h(t)$. It follows from equation (1) that:

$$\mathbf{r}^* - \mathbf{r} = \sum_{i=1}^{3} w_i(\mathbf{r}_i - \mathbf{r}_0)\left(\frac{f_i^*}{w^*} - \frac{f_i}{w}\right).$$

Having multiplied this equation scalarly by vector $\mathbf{e}$ and taking into account equations (2) and (3), conditions (4) and conditions $f_1^* = f_1$, $f_2^* = f_2$, $\mathbf{e} \cdot (\mathbf{r}_3 - \mathbf{r}_0) = 0$, we will obtain the following equation:

$$\Delta h = 2hw_1 t(1-t)\left(\frac{1}{w^*} - \frac{1}{w}\right) \ (\Delta h \geq 0), \tag{10}$$

where weight functions $w = w(t)$ and $w^* = w^*(t)$ are transformed to the form:

$$w(t) = 1 + 2t(1-t)(w_1 - 1), \ w^*(t) = w(t) - t(1-t)\left[(1-a_0^*)(1-t) + (1-a_3^*)t\right]. \tag{11}$$

Let us use inequality

$$(1-a_0^*)(1-t) + (1-a_3^*)t \leq \alpha\left[(1-t)^2 + t^2\right]^{1/2}, \tag{12}$$

where $\alpha = \left[(1-a_0^*)^2 + (1-a_3^*)^2\right]^{1/2}$.

Then from (10–12), we will obtain inequality $\Delta h \le 2\alpha w_1 hg$, where function $g = g(t)$ takes the following form:
$$g(t) = \frac{t^2(1-t)^2\left[(1-t)^2 + t^2\right]^{1/2}}{w(t)\left[w(t) - \alpha t(1-t)\left((1-t)^2 + t^2\right)^{1/2}\right]}$$

Research for the extremum of function $g(t)$ on interval [0, 1] shows that if $w_1 \le 3 + \sqrt{14}$ the function reaches its largest value when $t = 1/2$ at any values of a_0^* and a_3^* that satisfy conditions (9). In this case, the following inequality is obtained:
$$\Delta h \le \frac{\beta w_1 h}{(1 + w_1)(1 - \beta + w_1)}, \tag{13}$$

where parameter β is defined as $\beta = \alpha\sqrt{2}/4$. Note that the above limitation on weight w_1 is not essential, because of large values of weight w_1, segment C asymptotically approaches to the control polyline and this case is of no practical interest.

A more precise evaluation formula can be obtained when the following statement is used instead of inequality (12):
$$\left(1 - a_0^*\right)\cdot(1-t) + \left(1 - a_3^*\right)\cdot t = t\cdot(1-t)\cdot\left(2 - a_0^* - a_3^*\right) + (1-t)^2\left(1 - a_0^*\right) + t^2\left(1 - a_3^*\right) \le$$
$$t\cdot(1-t)\cdot(2 - a_0^* - a_3^*) + \alpha\left[(1-t)^4 + t^4\right]^{1/2}.$$

Then inequality (13) remains and parameter β takes the following form:
$$\beta = (2 - a_0^* - a_3^* + \alpha\sqrt{2})/8, \left(0 \le \beta \le 1/2\right).$$

In this case, evaluation is of sufficient precision. Numerical experiments show that its error is less than 20%. In the given example (figure 1), $\Delta h < 0.063$. The true deviation of segment L^* from segment L in the direction of vector **h** accurate to three decimal places is 0.060 (0.057 is perpendicular to segment L).

Comment 3. Let the deviation of some modified segment C^* from its source segment C exceed the acceptable deviation. Then segment C may be divided into two segments C_1 and C_2 [1] for the purpose of modification and so on. This algorithm is highly efficient for such problems as elimination of curvature discontinuities along a smooth compound curve, e.g. a curve that consists of circular arcs or biarcs. The algorithm should produce control polyline L_0 as an initial approximation, smooth curve L_1 as a first approximation and curve L_2 with continuous curvature that deviates from L_1 within the preset deviation.

4. Conclusion
The main objective of the paper was to demonstrate the potential [3] of automatic tuning of basis functions in combination with a posterior estimation of accuracy by the example of two problems. Formulation of the problems and their solutions can be generalized for the situation when the original curve is the non-planar generalized cone segment [1, 3, 16], presented in Boll basis. This allows a considerable improvement of the algorithms used for the problems of construction of cross section geometric objects. This is important, for example, in a CAD hot forging technology, where it is necessary to build several dozen cross sections for the calculation of the workpiece.

Nowadays, non-uniform rational B-splines (NURBS) [6] are widely used as a standard means to represent and design geometric objects, because they often lead to a more universal, simple and accurate decisions compared to traditional splines for modeling of complex curved surfaces. A crucial step forward here is to define node vectors (in addition to control points and their weights) as additional information.

Authors believe that still a more universal form of presentation and analytical description of geometric objects can be obtained if vectors parameters of basis functions are introduced.

This allows for transformation from Bernstein basis to other bases and back when solving several problems mentioned in [6], in particular, the problem of complicated conjugate surfaces.

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