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Contact interaction of flexible Timoshenko beams with small deflections

I V Papkova[1], A V Krysko[1,2], O A Saltykova[1,2] , A A Zakharova[2], V A Krysko[1]

[1]Department of Mathematic and modelling, Saratov State Technical University, Politehnicheskaya str, 77, Saratov, Russia
[2]Department of engineering graphics and industrial design, Tomsk Polytechnic University, Sovetskaya str, 84/3, Tomsk, Russia

Abstract. In this work chaotic dynamics contact interaction of two flexible Tymoshenko beams, under the action of a transversal alternating load is investigated. The contact interaction of the beams is taken into account by the Kantor model. The geometric nonlinearity is taken into account by the model of T. von Karman.The system of partial differential equations of the twelfth order reduces to the system of ordinary differential equations by the method of finite differences of the second order. The resulting system by methods of Runge-Kutta type of the second, fourth and eighth orders was solved.Our theoretical/numerical analysis is supported by methods of nonlinear dynamics and the qualitative theory of differential equations. Chaotic vibrations of two flexible beams of Timoshenko were investigated and the optimal step values over the spatial coordinate and the time steps for the numerical experiment were found. Convergence for all applicable numerical methods have been achieved and shown that chaotic signals are true.

Keywords: Contact interaction, Timoshenko's beam, chaos, finite difference method, Runge-Kutta type methods, geometric nonlinearity

1. Introduction

Although nonlinear dynamics and contact interaction of beams based on hypothesis of different approximations have been successively used for many years, there still are present open problems originated from both engineering and science. Several hypotheses are known that describe the dynamics of a beam element: first, second, third approximation theories, and others. The hypothesis of the first approximation is the Euler-Bernoulli hypothesis [1], the second-approximation hypothesis or the Tymoshenko hypothesis [2], allows us to take into account the normal rotation to the midline after deformation. This theory allows more accurate description of the dynamics of the beam element. The nonlinear dynamics of beams, based on various hypotheses, is investigated in articles [3-5]. This article allows to prove the truth of the chaotic vibrations of two beams of Timoshenko with a small gap, under action of the transverse alternating load, taking into account the contact interaction. In the known literature, there are no solutions to such problems. This is a fundamentally important issue. When solving such complex non-linear systems of equations by numerical methods, there is a probability of obtaining an incorrect solution due to errors in numerical methods. In this paper we shall use the definition of chaos given by Gulik [6]. Gulik believes that chaos exists when either there is a significant dependence on the initial conditions or the function has a positive Lyapunov exponent at each point of the region. In this case, it is not periodic. As initial conditions, in addition to the conditions imposed on the functions entering into the system of differential equations, we mean the number of partitions with respect to the spatial coordinate, the order of the Runge-Kutta method, and the kinematic hypothesis. To prove the reliability and truth of solutions in chaos methods of analysis of nonlinear dynamics were used. These methods include the construction of phase portraits, Fourier power spectra, Poincaré pseudo-mappings, wavelet spectra, signals, calculation of the value of the highest Lyapunov exponent in three different algorithms Kantz [7], Wolff [8] and Rosenstein [9]. If the results obtained by all these methods gave the same result, then we can assume that the results obtained are true. In addition, the question of the convergence of the results, depending on the number of partitions along the spatial coordinate and on the time step for the finite-difference method, has been investigated. In contrast to previous studies [10], in this paper it was required to achieve convergence of results in chaos not only in Fourier power spectra, but also in signals. A separate point of the study was the choice of the Runge-Kutta method for solving the contact problem.

2. Mathematical model

We study a two-layer beam, where the layers can contact each other as it is shown in Figure 1. We consider the case when sticking between the layers of the beam is not possible because contact pressure is small. The Cartesian Coordinate system introduced on the Fig. 1. The equations of motion of the beams, as well as the boundary and initial conditions, are obtained from energy principle of the Hamilton-Ostrogradskiy $\int_{t_0}^{t_1}(\delta K - \delta\Pi + \delta' W)dt = 0$, where K – is the kinetic energy, Π – is the potential energy, $\delta' W$ - is the sum of elementary work of external forces. In this coordinate system, a structure of two beams, like a two-dimensional domain Ω, is defined as follows $\Omega = \{x \in [0, a\} - h \le z \le h_k + 3h\}$, $0 \le t \le \infty$.

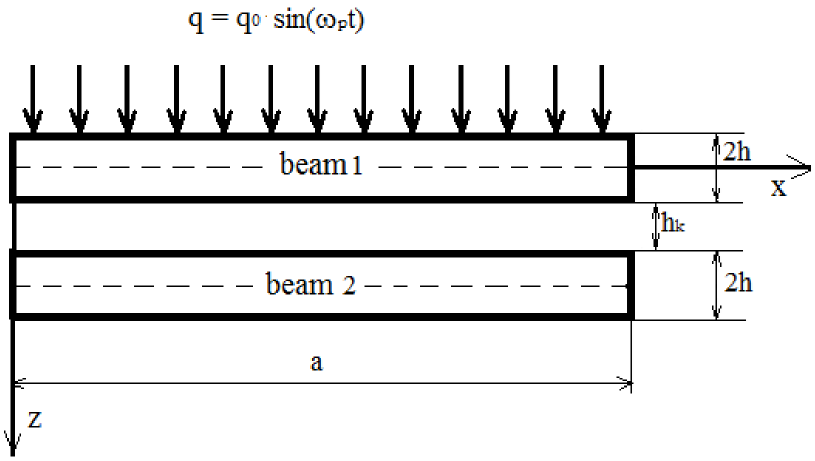


Figure 1. Scheme of the analyzed beam structure.

To simulate the contact interaction of beams according to the model of Kantor B.Ya. in the equations of the deflection beams, it is necessary to introduce the term $(-1)^i K(w_1 - w_2 - h_k)\Psi$, $i = 1,2$ - is a beam number, function Ψ is defined by $\Psi = \frac{1}{2}[1 + sign(w_1 - h_k - w_2)]$, where $\Psi = 1$, if there is contact between the beams - $w_1 > w_2 + h_k$, otherwise no contact [11], K – is a coefficient of rigidity of transverse compression of the structure in zone of the contact , h_k – is a gap between the beams. Tangential displacement u^z, w^z are distributed on thickness $\{-h \le z \le h\}$ by linear law $u_z = u + z\gamma_x$; $w_z = w$, where $\gamma_x = \gamma_x(x,t)$ — angle of rotation of the normal to the line $z = 0$. Then $e_{xx}^z = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \gamma_x}{\partial x}$; $e_{xz}^z = \gamma_x + \frac{\partial w}{\partial x}$ and if denoted by ε_{11} - tangential deformations of the midline, bending deformation through $H_{11} = \frac{\partial \gamma_x}{\partial x}$, deformation of shear $\varepsilon_{13} = \gamma_x + \frac{\partial w}{\partial x}$, then the expressions for the deformations will be represented in form of a linear expansion by degrees z: $e_{xx}^z = \varepsilon_{11} + zH_{11}$; $e_{xz}^z = \varepsilon_{13}$; $z \in (\delta_i - \Delta, \delta_{i+1} - \Delta)$. We introduce a shear deformation by the formula: $\varepsilon_{13}^z = \frac{1}{2hG_{13}} Q_1 f(z)$, where Q_1 - transverse force; G_{13} - shear modulus, $f(z)$ - function characterizing the law of distribution of tangential stresses along the thickness. Here $Q_1 = 2hG_{13}k^2\varepsilon_{13}$, where $\frac{1}{k^2} = \frac{1}{2h}\int_{-h}^{h} f^2(z)dz$ .Value k^2 for such a function will be equal to $\frac{5}{6}$.

Taking into account Hooke's law, the expression for the stresses can be written in the form $\sigma_{xx}^i = E^i\varepsilon_{11}^i + E^i z H_{11}^i$, $\sigma_{xz}^i = G_{13}^i \varepsilon_{13}^i$. Then $T_{11}^i = \int_{-h}^{h} \sigma_{xx}^i dz$ — is an internal efforts; $Q_{11}^i = \int_{-h}^{h} \sigma_{xz}^i dz$ — is a cutting forces; $M_{11}^i = \int_{-h}^{h} \sigma_{xx}^i z dz$ — is a bending moments. We write down the Hamilton-Ostrogradskiy principle $\delta S^i = \delta\int_{t_0}^{t} (K^i - \Pi^i)dt = 0$, where

$\Pi^i = \Pi_c^i + \Pi_u^i$. The energy of the middle surface is $\Pi_c^i = \frac{1}{2}\int_e T_{11}^i \varepsilon_{11}^i dx = \frac{1}{2}\int_e T_{11}^i \left(\frac{\partial u_i}{\partial t} + \frac{1}{2}\left(\frac{\partial w_i}{\partial t}\right)^2\right)dx$. The bending energy is expressed as follows $\Pi_u^i = \frac{1}{2}\int_e \left[M_{11}^i H_{11}^i + Q_{11}^i\left(\gamma_{xi} + \frac{\partial w_i}{\partial t}\right)\right]dx$, but the kinetic energy $K^i = \frac{1}{2}\frac{(2h)\gamma_i}{g_i}\int_e \left[\left(\frac{\partial u_i}{\partial t}\right)^2 + \left(\frac{\partial w_i}{\partial t}\right)^2 + \left(\frac{\partial \gamma_{xi}}{\partial t}\right)^2\right]dx$.

From the Hamilton-Ostrogradskiy principle, we write the equations of motion of a structure of two Timoshenko beams in displacements taking energy dissipation into account in the dimensionless form as follows:

$$\begin{cases} \frac{1}{3}\left(\frac{\partial^2 w_i}{\partial x^2} + \frac{\partial \gamma_{xi}}{\partial x}\right) + \frac{1}{\lambda^2}\left(L_1(w_i,u_i) + \frac{3}{2}L_2(w_i,w_i) + L_3(w_i,u_i)\right) + \\ + (-1)^i K(w_1 - w_2 - h_k)\Psi + q(t) - \frac{\partial^2 w_i}{\partial t^2} - \varepsilon_1 \frac{\partial w_i}{\partial t} = 0; \\ \frac{\partial^2 u_i}{\partial x^2} + L_4(w_i,w_i) - \frac{\partial^2 u_i}{\partial t^2} = 0; \\ \frac{\partial^2 \gamma_{xi}}{\partial x^2} - 8\lambda^2\left(\frac{\partial w_i}{\partial x} + \gamma_{xi}\right) - \frac{\partial^2 \gamma_{xi}}{\partial t^2} = 0;\ i = 1,2, \end{cases} \tag{1}$$

Here $i = 1,2$ - is a serial number beams, $L_1(w_i,u_i) = \frac{\partial^2 w_i}{\partial x^2}\frac{\partial u_i}{\partial x}$, $L_2(w_i,w_i) = \frac{\partial^2 w_i}{\partial x^2}\left(\frac{\partial w_i}{\partial x}\right)^2$, $L_3(w_i,u_i) = \frac{\partial w_i}{\partial x}\frac{\partial^2 u_i}{\partial x^2}$, $L_4(w_i,w_i) = \frac{\partial w_i}{\partial x}\frac{\partial^2 w_i}{\partial x^2}$ - are the non-linear operators, γ_{xi} - is a function transverse shear, w_i, u_i – are the functions of deflections and displacements of beams, respectively. To the system of differential equations (1), we must add boundary conditions and initial conditions.

The system of equations (1), the boundary and initial conditions are reduced to the dimensionless form by means of variables:

$$\bar{w} = \frac{w}{2h},\ \bar{u} = \frac{ua}{(2h)^2},\ \bar{x} = \frac{x}{a},\ \lambda = \frac{a}{(2h)},\ \bar{q} = q\frac{a^4(1-\nu^2)}{(2h)^4 E},$$

$$\bar{t} = \frac{t}{\tau},\ \tau = \frac{a}{c},\ c = \sqrt{\frac{Eg}{(1-\nu^2)\rho}},\ \bar{\varepsilon}_1 = \varepsilon_1\frac{a}{c},\ \bar{\gamma}_x = \frac{\gamma_x a}{(2h)}.$$

The resulting system of nonlinear partial differential equations (1) together with the boundary and initial conditions reduces to the system of ordinary differential equations by the finite differences method with approximation $O(c^2)$, where c – step in the spatial coordinate. At each point of the grid, we obtain a system of ordinary differential equations. The Cauchy problem obtained in time is solved by methods of the Runge-Kutta type. In the work, different Runge-Kutta methods are compared: the Runge-Kutta of the 4th (rk4), 2nd (rk2) orders, the Runge-Kutta-Felberg 4th order (rkf45), 4th order Kesh-Karp (rkck), the Runge-Kutta Prince-Dormand of the eighth order (rk8pd), the implicit Runge-Kutta method of the second order (rk2imp) and the 4th order (rk4imp). On basis of this algorithm was created programs, which allow to solve the problem depending on the control parameters $\{q_0, \omega_p\}$. Great attention in work was given to the question of not penetrating the elements of the structure into each other. As noted above, the problems under study are highly nonlinear, so the question arises about the reliability of the results obtained.

3. Numerical results

The boundary conditions for sealing both ends of the beams:

$$w_i(0,t) = w_i(1,t) = u_i(0,t) = u_i(1,t) = \frac{\partial w_i(0,t)}{\partial x} = \frac{\partial w_i(1,t)}{\partial x} = 0, i = 1, 2$$

Initial conditions:

$$w_i(x,t)_{|t=0} = u_i(x,t)_{|t=0} = \gamma_{xi}(x,t)_{|t=0} = 0,\ \frac{\partial w_i(x,t)}{\partial t}_{|t=0} = \frac{\partial u_i(x,t)}{\partial t}_{|t=0} = \frac{\partial \gamma_{xi}(x,t)}{\partial t}_{|t=0} = 0.$$

On the first beam is acted transverse distributed over the surface sign-variable load type: $q = q_0 \sin(\omega_p t)$, where q_0 - amplitude, ω_p - frequency of driving vibrations.

For the numerical experiment we consider: $\omega_p = 5.1$, $q_0 = 5000$, $h_k = 0.1$, $\lambda = a/2h = 50$, $\varepsilon_1 = 1$. The frequency of the driving vibrations is close to the natural frequency of the beam. A preliminary question was investigated about convergence of the finite differences method. In Fig. 2 (a, б) signals are presented, calculated for a different number of points division of a segment $n = 40; 80; 120; 240; 360; 400; 440$. For the second beam, the convergence by the number of divisions of the segment is much worse and does not completely come. The error between signals calculated at n = 360 и n = 400 isc 3%, However, the signals coincide in shape over the entire time interval. The results were obtained using the Runge-Kutta method of the 8th order in the Prince-Dormand modification (rk8pd).

a) b)

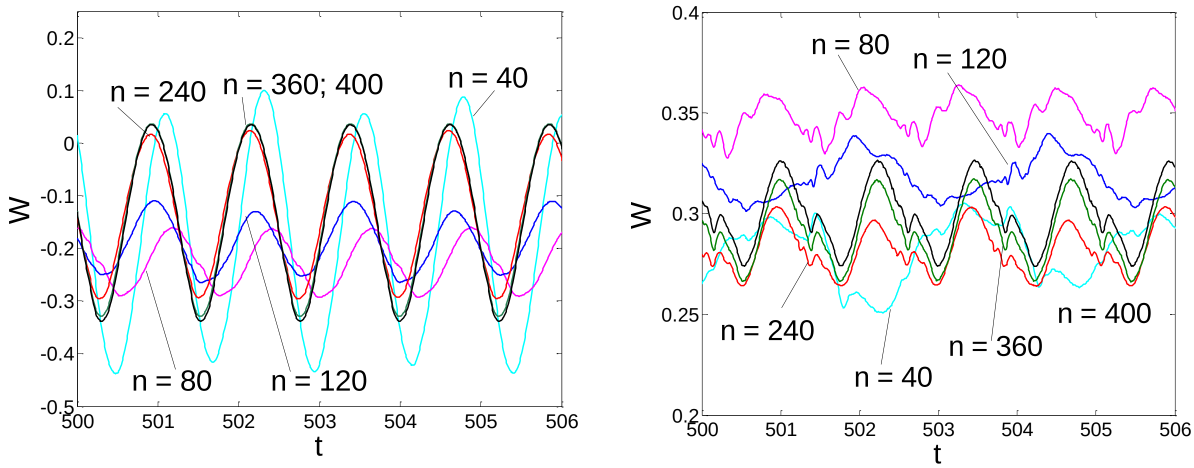


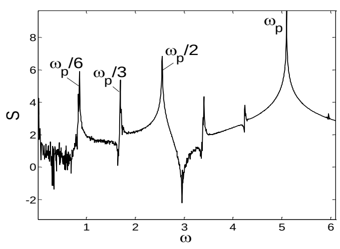
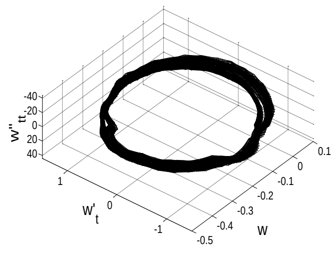
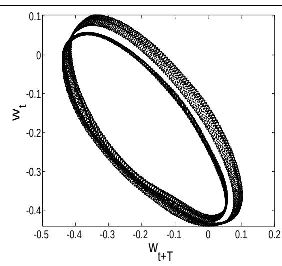
Figure 2. Beam deflection of signals $w_i(0.5, t)$ at $t \in [500..506]$ for $n = 40; 360; 400$ a) First beam, b) Second beam.

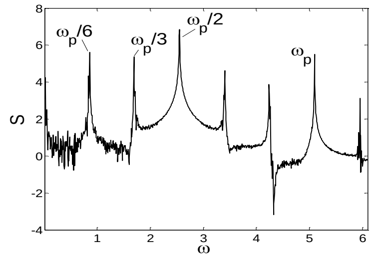
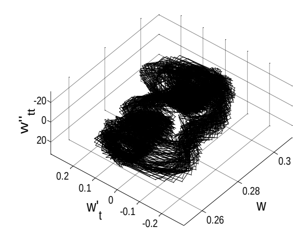
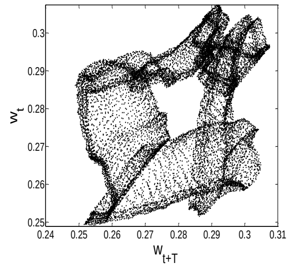
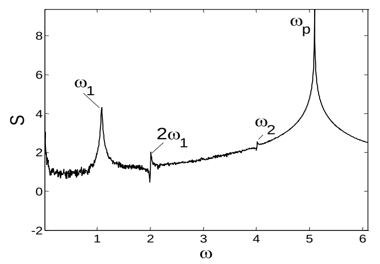
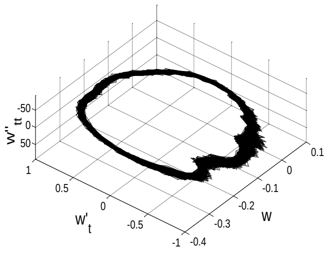
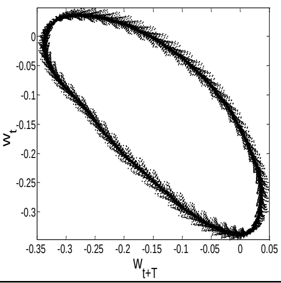
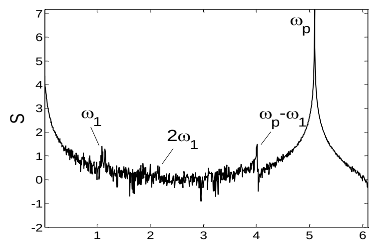
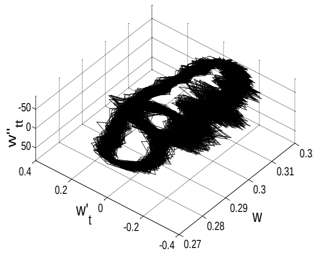
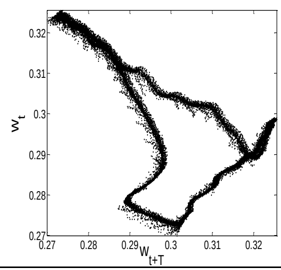
In [10], it was considered sufficient to show convergence with respect to the Fourier power spectra for chaotic vibrations. By signal, it was impossible to achieve convergence. At $n = 400$ the convergence of results for chaotic results, even with a signal, will be achieved.

Further, the convergence of the signals was investigated depending on the type of the Runge-Kutta method. For both beams, the results of the Runge-Kutta methods of the second, fourth and eighth orders coincided completely, however, it was decided to use the Prince-Dormand 8-th order method (rk8pd) in further calculations, since this method allows automatic step-by time.

Let us investigate the dynamic characteristics of beams for different number of partitions with respect to the spatial coordinate. In Table 1 we give graphs of Fourier power spectra, 3D phase portraits, and Poincaré pseudo-mappings for both beams.

Table 1. Dynamic characteristics of beams

n	№	Power spectrum	Phase portrait 3D $w(w'_t, w''_{tt})$	Pseudo Poincare map
40	1			

	2			
360,400	1			
	2			

4. The discussion of the results.

Analyzing the Fourier power spectra for different numbers of partitions n with respect to the spatial coordinate, for beam 1, we can first note an increase, and then a reduction in the number of frequencies and a reduction in the noise component with n. At $n = 40$ the power spectra of both beams demonstrate vibrations at linearly dependent frequencies $\omega_p / 2$, $\omega_p / 3$, $\omega_p / 6$ and the presence of a noise component at low frequencies. Both beams vibrate at the same frequencies, i.e. frequency synchronization of vibrations occurs. An increase in n doubles the overall chaotic pedestal. At $n = 120$ in the power spectrum of both beams, the frequency again appears $\omega_p / 2$, also a decrease in the chaotic component as compared to $n = 80$. At $n = 240$ in the signal of the first and second beams there are frequencies в $\omega_p / 2$ и $\omega_p / 4$. При $n = 360$ и $n = 400$ power spectra are cleaned from the noise component, in the signal of both beams there are frequencies $\omega_1 = \omega_p / 5 = 1.02$, $2\omega_1$ и $\omega_2 = \omega_p - \omega_1$, then we can talk about synchronization of oscillations at these frequencies.

As a rule, 2D phase portraits are considered, but in this paper it is proposed to consider 3D phase portraits $w(w'_t,\ w''_{tt})$, in this case we will have information about all the characteristics of the dynamics. Consider 3D phase portraits $w(w'_t,\ w''_{tt})$. At the minimum number of nodes n the phase portrait for beam 1 gives a ring, but in space it is clear that this ring has a thickness and is non-uniform. For the second beam, the phase portrait is a solid spot, which corresponds to chaotic vibrations. With an increase in the number of equations, the phase portrait and the appearance of rings are compressed.

The pseudo-Poincare map for a beam 1 for all n has the shape of an oval, but with small changes in its thickness. Beginning from n = 360 their convergence is noted, as well as for beam 2, where for n = 40 the pseudo-Poincaré map is scattered.

Complex analysis of frequency characteristics, it allows us to make the conclusion about the sufficiency n = 400, for the study of nonlinear dynamics and the contact interaction of two beams with a gap, when both beams are described by Timoshenko's model. In this paper, the values of the highest Lyapunov exponent for n = 400 are calculated, using the methods Wolff, Rosenstein and Kantz. The values are obtained on the basis of solutions of the Cauchy problem by the Runge-Kutta method of the 8th order (rk8pd). Different methods of calculating Lyapunov's indices must be used to determine the true chaos. When the number of points of division of the beam

on n = 40; 80, 120; 240, 360, 400 segments in the method of finite differences, the Lyapunov exponents converge at any counting method to the second or third decimal place.
In Table 2, we give the values of the highest Lyapunov exponent for both beams at n = 400, calculated using the methods of Kantz, Wolf, and Rosenstein. In order to avoid obtaining erroneous conclusions in the study of chaotic oscillations, the Lyapunov exponents are obtained by several methods. At present there is no reliable method for determining such.

Table 2. Lyapunov exponent

n		**Beam 1**			**Beam 2**		
	method	Wolf	Rosenstein	Kantz	Wolf	Rosenstein	Kantz
n = 400	Rk8pd	0.01658	0.05646	0.02191	0.02835	0.04617	0.02363

All the values of the highest Lyapunov exponent, regardless of the method of solving the Cauchy problem, from the number of intervals of the beam partition, from the calculation algorithm are positive. That is, we are dealing with the true chaotic oscillations of the investigated beam structure.

5. Conclusions

In this article the reliability of the numerical results of the solution for the problem of the two beams contact interaction, described by the kinematic hypothesis of Tymoshenko, with a small gap between them, was given and defended. A comprehensive study of the nonlinear dynamics of the contact interaction of Timoshenko beams under action of a transversal alternating load. Selection number of points of partitions on the spatial coordinate (n = 400) and the choice of the method for solving the Cauchy problem (the Runge-Kutta method of the 8th order of Prince-Dormand) was justified. On basis of carried out research of frequency characteristics, we can talk about the phenomenon of chaotic frequency synchronization of beams vibration.

6. Acknowledgments

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