NONLOCAL ONE-DIMENSIONAL FISHER–KOLMOGOROV–PETROVSKII–PISKUNOV EQUATION WITH ABNORMAL DIFFUSION

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Abstract. Analytical solutions are constructed for the nonlocal space fractional Fisher–Kolmogorov–Petrovskii–Piskunov equation with abnormaldiffusion. Such solutions allow us to describe quasi-steady state patterns. Special attention is given to the role of fractional derivative. Fractional diffusion equations are useful for applications in which a cloud of particles spreads faster than predicted by the classical equation. The resulting solutions spread faster than the classical solutions and may exhibit asymmetry, depending on the fractional derivative used. Results of numerical simulations and properties of analytical solutions are presented. Influence of the fractional derivative on patterns ordered in space and time is discussed.

Keywords:Fractional reaction–diffusion, pattern formation, nonlocal population dynamics, Fisher–Kolmogorov–Petrovskii–Piskunov equation, semiclassical approximation.

INTRODUCTION

Reaction–diffusion equations are useful in many areas of science and engineering [1]. In applications to population biology, the reaction term models growth, and the diffusion term accounts for migration .The classical diffusion term originates from a model in physics. Recent research indicates that the classical diffusion equation is inadequate to model many real situations, where a particle plume spreads faster than that predicted by the classical model, and may exhibit significant asymmetry. These situations are called anomalous diffusion [2] . One popular model for anomalous diffusion is the fractional diffusion equation, where the usual second derivative in space is replaced by a fractional derivative of order $0<\alpha<2$. Solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation, and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

Nonlocal reaction-diffusion (RD) models are generally used to describe structures ordered in space and time. Structures of this type, formed by self-organization mechanisms, are involved in many important phenomena in biology, medicine, epidemiology, and ecology, such as the pattern formation in population dynamics, cancer treatment, evolution of infectious diseases, etc. (see, e.g., the review papers [3, 4], and references therein). The evolution of one-species microbial populations with long-range interactions between individuals is modeled by a nonlocal generalization of the classical Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equation [3, 4] for population density u(x,t):

$$u_t(x,t) = D\Delta u(x,t) + au(x,t) - bu^2(x,t). \qquad (1)$$

Equation (1) contains terms that describe a diffusion process with coefficient D, population growth with rate a, and the local competition between individuals with rate b.

Nonlocal effects arise in competitive interactions of microbial populations due to the diffusion of nutrients, the release of toxic substances, chemo taxis, and molecular interactions between individuals.

In generalized FKPP equation local quadratic losses $bu^2(x,t)$ are replaced by an integral expression $u(x,t)\int b_\gamma(x,y)u(y,t)dy,$ which takes into account nonlocal interactions in the population through the influence function $b_\gamma(x,y)$. Parameter γ describes the effective area of interaction between individuals in population. So, when $\gamma \to 0$ it's fair $b_\gamma(x,y) \to b\delta(x-y)$, and nonlocal losses go to local $bu^2(x,t)$ FKPP equation considering nonlocal quadratic losses in the interval $[-l,l]$ will be

$$u_t(x,t) = Du_{xx}(x,t) + au(x,t) - \chi u(x,t)\int_{-l}^{l} b_\gamma(x,y)u(y,t)dy. \quad (2)$$

Spatio- time structures (patterns) are not formed in the course of evolution , described by the classical FKPP equation (1). Nonlocal FKPP equation allows to describe the formation of structures that arise due to the loss of competitive and non-local diffusion with an appropriate choice of parameters of the equation.

Note that the main method of examination of structure formation in above works is numerical simulation . The paper is focused on the analytical method. One-dimensional model is chosen for mathematical simplicity .

DEFINATION OF FRACTIONAL DERIVATIVES

There are several different approaches to the definition of fractional order derivative, reflecting the peculiarities of fractional calculus. The most widely and frequently used definition is made by Riemann-Liouville, based on the generalization of the Abel equation [5]

$$D_-^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^{x}\frac{f(t)dt}{(x-t)^{\alpha+1-n}}, \quad (n-1 < \mathrm{Re}(\alpha) \le n, n \in N)$$

Here the standard notation is used for the differentiation operator and Γ-functions.

Simplification of this definition is the definition made by Caputo, which is applicable for sufficiently smooth functions where the operation of differentiation may be included under the integral sign:

$$D_-^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{x}\frac{f^{(n)}(t)dt}{(x-t)^{\alpha+1-n}}, \quad (n-1 < \mathrm{Re}(\alpha) \le n, n \in N)$$

A.Gryunvald and independently Letnikov introduced the concept of fractional derivative as the limit of difference relations:

$$f^{(\alpha)}(x) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k C_n^k f(x+(n-k)h)$$

If $f(x)$ is continuous and $df(x)/dx$ is integrable in the interval $[a,x]$, then the derivatives of the Riemann-Liouville and Caputo and Grunvald-Letnikov exist and coincide.

ONE-DIMENSIONAL NONLOCAL FKPP EQUATION

Let's consider the equation (2) with a difference kernel $b_\gamma(x,y) = b_\gamma(x-y)$, $b_\gamma(x)$ is assumed to be even:

$$u_t(x,t) = Du_\alpha(x,t) + au(x,t) - \chi u(x,t)\int_{-l}^{l} b_\gamma(x-y)u(y,t)dy, \quad (3)$$

where α – is an order of fractional derivative. The functions $b_\gamma(x)$ and $u(x,t)$ are expanded in a Fourier series:

$$b_\gamma(x-y)=\sum_{m=-\infty}^{\infty} b_m e^{i\pi m(x-y)/l}, \qquad b_m=\frac{1}{2l}\int_{-l}^{l} b(\tau)e^{-i\pi m\tau/l}d\tau.$$

$$u(x,t)=\sum_{k=-\infty}^{\infty}\beta_k(t)e^{i\pi kx/l}, \qquad \beta_k(t)=\frac{1}{2l}\int_{-l}^{l}u(z,t)e^{-i\pi kz/l}dz. \qquad (4)$$

Then (4) $u_\alpha(x,t)$, according to [5], will be defined as

$$u_\alpha(x,t)=\sum_{k=-\infty}^{\infty}\left(\frac{ik\pi}{l}\right)^{\alpha}\beta_k(t)e^{i\pi kx/l}, \quad i^\alpha=e^{i\pi\alpha/2}.$$

The derivative of the exponent is e^{ax} with order $0<\alpha<1$ by defination of fractional derivative of Caputo

$$D_-^\alpha e^{ax}=\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{e^{at}dt}{(x-t)^\alpha}=\left|x-t=\tau\right|=\frac{e^{ax}}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{e^{-a\tau}dt}{\tau^\alpha}=$$

$$=\left|\begin{array}{l} a\tau=t \\ d\tau=\frac{dt}{a}\end{array}\right|=\frac{a^\alpha e^{ax}}{\Gamma(1-\alpha)}\int_0^\infty e^{-t}t^{(1-\alpha)-1}dt=a^\alpha e^{ax}$$

And by defination of Gryunvald-Letnikov

$$(e^{ax})^{(\alpha)}=\lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k C_\alpha^k e^{ax+a(\alpha-k)h}=$$

$$=e^{ax}\lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k C_\alpha^k e^{ah(\alpha-k)}=e^{ax}\lim_{h\to 0}\frac{1}{h^\alpha}(e^{ah}-1)^\alpha=a^\alpha e^{ax}$$

The differentiation with respect to time ratio (4), expressing the function $u_t(x,t)$ presented in (3), shows:

$$\dot\beta_k(t)=\frac{1}{2l}\int_{-l}^{l}e^{\frac{-i\pi kz}{l}}u_t(z,t)dz=\frac{1}{2l}\int_{-l}^{l}dz\left(Du_\alpha(z,t)+au(z,t)-\varkappa u\sum_{n=-\infty}^{\infty}b_n e^{\frac{i\pi nz}{l}}\beta_n(t)\right)e^{\frac{-i\pi kz}{l}}=$$

$$=\frac{1}{2l}\sum_{j=-\infty}^{\infty}\beta_j(t)\int_{-l}^{l}dz\left(D\left(\frac{ij\pi}{l}\right)^\alpha+a-\chi\sum_{n=-\infty}^{\infty}b_n e^{\frac{i\pi nz}{l}}\beta_n(t)\right)e^{\frac{i\pi z(j-k)}{l}}=$$

$$=\sum_{j=-\infty}^{\infty}\beta_j(t)\left[\frac{1}{2l}\left(D\left(\frac{ij\pi}{l}\right)^\alpha+a\right)\int_{-l}^{l}e^{\frac{i\pi z(j-k)}{l}}dz-\frac{\chi}{2l}\sum_{n=-\infty}^{\infty}\beta_n(t)b_n\int_{-l}^{l}e^{\frac{i\pi z(n-(k-j))}{l}}dz\right]=$$

$$=\sum_{j=-\infty}^{\infty}\beta_j(t)\left[\left(D\left(\frac{ij\pi}{l}\right)^\alpha+a\right)\delta_{j,k}-\chi\sum_{n=-\infty}^{\infty}\beta_n(t)b_n\delta_{n,k-j}\right].$$

[redacted] $\left(\frac{ik\pi}{l}\right)^\alpha+a\Big)-\chi\sum_{j=-\infty}^{\infty}\beta_{k-j}b_{k-j}\beta_j, \quad k=\overline{-\infty,\infty}.$ (5)

We will seek the coefficients β_j in the form $\beta_j(t)=\beta_0(t)\delta_{j0}$.

[redacted] $(t)a-\chi\beta_0^2(t)b_0.$ (6)

Equation (6), with the initial conditions $\beta_j|_{t=0}=\beta_{00}\delta_{j0}$, is

$$\beta_0(t)=\frac{\beta_{00}e^{at}}{1+\frac{\chi b_0\beta_{00}}{a}(e^{at}-1)}. \qquad (7)$$

Now let's look for solutions of the equation (5) β_j in the form

$$\beta_j(\theta,\tau,T) = \beta_j^{(0)}(\theta,\tau) + \frac{1}{T}\beta_j^{(1)}(\theta,\tau) + \text{[redacted]} = \phi(\tau)T, \quad (8)$$

where $\beta_j^{(0)}$ is defined by (7). Expansion (8) with (4) induces expansion

$$u(x,t) = u^{(0)}(x,t) + \frac{1}{T}u^{(1)}(x,t), \; (9)$$

Taking into the account the rules of differentiation of composite functions we get

$$\frac{d}{dt} = \frac{\partial\theta}{\partial t}\frac{\partial}{\partial\theta} + \frac{\partial\tau}{\partial t}\frac{\partial}{\partial\tau} = \varphi_\tau\frac{\partial}{\partial\theta} + \frac{1}{T}\frac{\partial}{\partial\tau}.$$

As a result, the system (5) takes the form

$$\left[\varphi_\tau\frac{\partial}{\partial\theta} + \frac{1}{T}\frac{\partial}{\partial\tau}\right]\left(\beta_j^{(0)} + \frac{1}{T}\beta_j^{(1)} + \ldots\right) = \bar{a}_j\left(\beta_j^{(0)} + \frac{1}{T}\beta_j^{(1)} + \ldots\right) - \chi\sum_{p=-\infty}^{\infty} b_p\left(\beta_{j-p}^{(0)} + \frac{1}{T}\beta_{j-p}^{(1)} + \ldots\right)\left(\beta_p^{(0)} + \frac{1}{T}\beta_p^{(1)} + \ldots\right),$$

where
$$\bar{a}_j = \left(D\left(\frac{ij\pi}{l}\right)^{\alpha} + a\right).$$

Equating terms of the same power $1/T$, we obtain

$$\varphi_\tau\frac{\partial}{\partial\theta}\beta_j^{(0)} = \bar{a}_j\beta_j^{(0)} - \chi\sum_{p=-\infty}^{\infty} b_p\beta_{j-p}^{(0)}\beta_p^{(0)}, \quad \text{и} \quad \varphi_\tau\frac{\partial}{\partial\theta}\beta_j^{(1)} = \bar{a}_j\beta_j^{(1)} - \chi\sum_{p=-\infty}^{\infty} b_p(\beta_{j-p}^{(1)}\beta_p^{(0)} + \beta_{j-p}^{(0)}\beta_p^{(1)}) - \frac{\partial}{\partial\tau}\beta_j^{(0)}, \text{[redacted]} (10)$$

Let's $\varphi(\tau) = a\tau$. Then from (10) it follows that

$$\frac{\partial}{\partial\theta}\beta_0^{(1)} = \beta_0^{(1)} - \frac{2\chi b_0}{a}\beta_0^{(0)}\beta_0^{(1)}, \; \frac{\partial}{\partial\theta}\beta_j^{(1)} = \frac{\bar{a}_j}{a}\beta_j^{(1)} - \frac{\chi}{a}\beta_0^{(0)}(b_0\beta_j^{(1)} + b_j\beta_j^{(1)}). \; (11)$$

Solving the system (10) and (11), considering that $\frac{\partial}{\partial\tau}\beta_j^{(0)} = 0$, we will find the coefficients $\beta_j^{(0)}$ and $\beta_j^{(1)}$. For the case of symmetric initial distribution up to $O(1/T^2)$ we obtain

$$u(x,t) = \frac{\beta_{00}e^{at}}{1 + \frac{\chi b_0\beta_{00}}{a}(e^{at} - 1)} + \frac{1}{T}\sum_{j=-\infty}^{\infty}\frac{\beta_{1l}e^{\text{[redacted]}}e^{ij\pi x/l}}{\left[1 + \frac{\chi\lambda_0\beta_{00}}{a}(e^{at} - 1)\right]^{(b_j + b_0)/b_0}}.$$

$$\operatorname{Im}\left(u^{(1)}(x,t)\right) = \sum_{j=-\infty}^{\infty}\frac{e^{\left(D\left|\frac{j\pi}{l}\right|^{\alpha}\cos\left(\frac{\pi}{2}\alpha\right) + a\right)t}\sin\left[D\left|\frac{j\pi}{l}\right|^{\alpha}\sin\left(\frac{\pi}{2}\alpha\operatorname{sgn}j\right)t + \frac{j\pi x}{l}\right]}{\left(1 + \frac{\varkappa\lambda_0\beta_{00}}{a}(e^{at} - 1)\right)^{(b_j + b_0)/b_0}}.$$

You may notice that $\operatorname{Im}\left(u^{(1)}(x,t)\right) = 0$. Let's choose $b_\gamma = b_0\exp\left\{-(x-y)^2/\gamma^2\right\}$.

Now let's consider how the population density depends on the degree of diffusion.

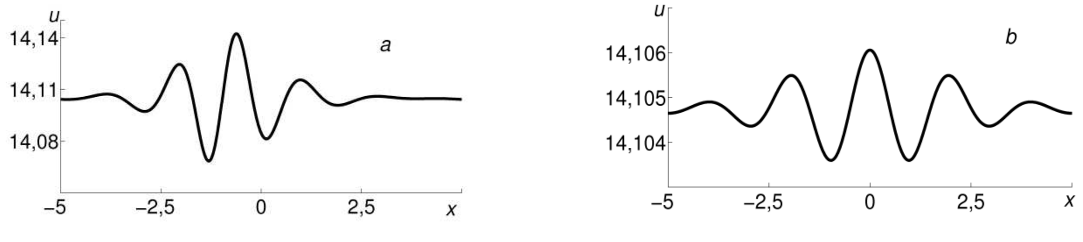

Fig. 1. Graph of function $u(x,t)$ for $t = 50$ and $\alpha = 1.5$ (a), 2 (b), $a = 0.5$, $b_0 = \gamma = 1$, $\chi = 0.2$, $D = 0.01$, $l = 5$, $T = 10$

The first and second moments $u(x,t)$ for $\alpha = 1.5$, respectively calculated by the formulas (fig. 2):

$$M(t) = \int_{-l}^{l} xu(x,t)dx, \qquad D(t) = \int_{-l}^{l} (x - M(t))^2 u(x,t)dx.$$

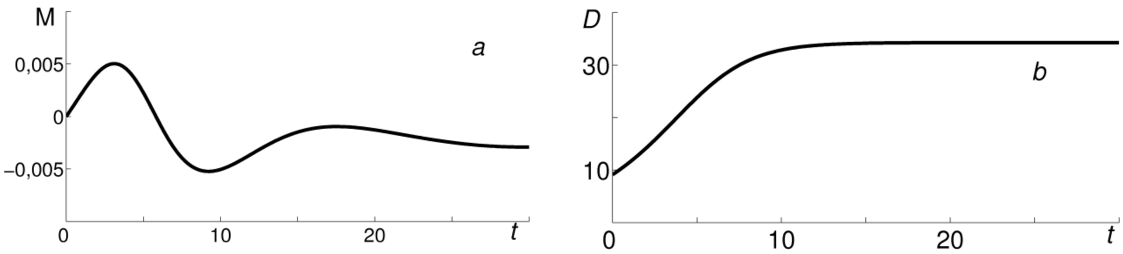

Fig. 2. Graph of the first initial moment $M(t)$ (a) and second central moment $D(t)$ (b).

As can be seen from the graphs (fig. 1), the lower the order of the fractional derivative, the greater the displacement of the center and stronger the deviation from the steady state.

CONCLUSION

The phenomenon of pattern formation in one-species populations was studied using a number of models based on generalized Fisher–Kolmogorov–Petrovskii–Piskunov (FKPP) equations taking into account nonlocal interaction effects.The paper has been focused on a special type of pattern formation with abnormal diffusion. The lower the order of the fractional derivative, the greater the displacement of the center and stronger the deviation from the steady state.

This solution is spatially homogeneous and monotonically depending on time. By analogy with previous studies, it was assumed that the patterns above can be described as large time perturbations of this exact solution. The large time asymptotics are constructed explicitly, to within $O(1/T^2)$, in the class of functions which tend to the above exact solution as $T \to \infty$. Thereby, the exact solution can be regarded as an attractor of the constructed class of asymptotic solutions and, hence, of the corresponding concentrated patterns. As the patterns evolve monotonically without qualitative changes to some steady-state, it's concluded asymptotic solutions describe approximately the quasi-steady-state patterns. The contribution of diffusion to the pattern formation has been investigated.

The approach used allows one, on the one hand, to gain information on the most essential characteristics of patterns and, on the other hand, to apply the methods developed for 1D problems to multidimensional problems.

The formalism proposed can be generalized to concentration manifolds of more general topological structure, such as multiply connected manifolds, and to curved manifolds describing the growth of microbial populations on complex structure objects.

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